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MDQ/D-21 **5014**

MATHEMATICS
MM-501
Functional Analysis

Time : Three Hours] [Maximum Marks : 80

Note : Attempt *Five* questions in all, selecting *one* question from each Section and the compulsory question.

Section I

1. (a) Show that C[a, b] can be made into an incomplete normal linear space w.r.t. a suitable norm. **8**

(b) In a finite dimensional normed space the closed unit ball is compact. What can you say about the converse of this statement ? Give full justification. **8**

2. (a) Discuss the boundedness of : **8**

(i) Differentiation operator

(ii) Integral operator.

(b) Compute the dual space of the space co. **8**

Section II

3. (a) Let X be a normed space and let $x_0 \neq 0$ be any element of X. Show that there is a bounded linear functional $\hat{f}$ on X such that : **8**

$$\|\hat{f}\| = \|x_0\|^{-1} \text{ and } \hat{f}(x_0) = 1.$$

(b) A reflexive normed space is a Banach space. Prove or disprove this statement. **8**

4. (a) If the dual space X′ of a normed space X is separable, show that X itself is separable. **8**

(b) If X and Y are Banach spaces and $T_n \in B(X, Y)$, $n = 1, 2, \ldots$, show that the following statements are equivalent : **8**

(i) $(\|T_n\|)$ is bounded

(ii) $(\|T_n x\|)$ is bounded for all $x \in X$,

(iii) $(|g(T_n x)|)$ is bounded for all $x \in X$ and all $g \in Y'$.

Section III

5. (a) (i) Show that $T : \mathbb{R}^2 \to \mathbb{R}$ defined by $(\xi_1, \xi_2) \mapsto (\xi_1)$ is open. Is the mapping $(\xi_1, \xi_2) \mapsto (\xi_1, 0)$ an open mapping ?

(ii) Show that an open mapping need not map closed sets onto closed sets. **8**

(b) Let X and Y be Banach spaces and T : D(T) $\to$ Y a closed linear operator, where D(T) $\subset$ X. If D(T) is closed in X, show that the operator T is bounded. **8**

6. (a) Let X be the vector space of all ordered pairs of complex numbers. Can we obtain the norm defined on X by $\|x\| = \|\xi_1\| + |\xi_2|$, $x = (\xi_1, \xi_2)$ from an inner product ? Justify. **6**

(b) Show that the subset $M = \{y = (\eta i) : \sum n_j = 1\}$ of complex space $\mathbb{C}^n$ is complete and convex. Also find the vector of minimum norm in M. **10**

Section IV

7. (a) Let (e_k) be an orthonomal sequence in a Hilbert space H. Show that for every $x \in$ H, the vector $y = \sum_{k=1}^{\infty} < x, e_k > e_k$ exists in H and $x - y$ is orthogonal to every e_k. **8**