Roll No. .... **Total Pages : 05**

MDE/D-21 **4315**

REAL ANALYSIS-I

MM-402

Time : Three Hours] [Maximum Marks : 80

Note : Attempt *Five* questions in all, selecting *one* question from each Section and the compulsory question.

Section I

1. (a) Show that $g \in R(\alpha)$ on $[a, b]$ if and only if for every $\varepsilon > 0$, there exists a partition Q of $[a, b]$ such that $U(\theta, g, \alpha) - L(\theta, g, \alpha) < \varepsilon$. **8**

(b) Suppose $f \geq 0$, f is continuous on $[a, b]$ and $\int_a^b f(x)dx = 0$. Prove that $f(x) = 0$ for all $x \in [a, b]$. State clearly the results used by you. **8**

2. (a) If g maps $[a, b]$ into $\mathbf{R}^n$ and if $g \in R(\alpha)$ for some monotonically incrasing function α on $[a, b]$, then show that $|g| \in R(\alpha)$, and $\left|\int_a^b g\, d\alpha\right| \leq \int_a^b |g| d\alpha$. **8**

(b) Let r_1, r_2, r_3 be curves in the complex plane, defined on [0, 2π] by $r_1(t) = e^{2it}, r_2(t) = e^{2\pi it\sin(1/t)}$, $r_3(t) = e^{it}$. Check which of these curves are rectifiable, and find their lengths. **8**

Section II

3. (a) Test for uniform convergence :

(i) The sequence $\{f_n\}$ defined by

$$f_n(x) = \frac{n^2 x}{1+n^3x^2},\ x \in [0, 1]$$ **5**

(ii) The series : **5**

$$\frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8} + \ldots\ldots,\ -\frac{1}{2} \le x \le \frac{1}{2}$$

(b) Let $f_n(x) = \dfrac{x^2}{x^2 + (1-nx)^2}\ (0 \le x \le 1, n = 1, 2, 3, \ldots\ldots)$.

Show that no subsequence of $\{f_n\}$ can convergence uniformly on [0, 1]. **6**

4. (a) For every interval [–*a*, *a*], show that there is a sequence of real polynomials P_n such that $P_n(0) = 0$ and such that $\lim\limits_{x\to\infty} P_n(x) = |x|$ uniformly on [–*a*, *a*].

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(b) If f is continuous on [0, 1] and if $\int_0^1 f(x)x^n dx = 0, n = 0, 1, 2, ...,$ then show that $f(x) = 0$ for each x in [0, 1]. **6**

Section III

5. (a) Suppose E is an open subset of $\mathbf{R}^n$ and f maps E into $\mathbf{R}^n$. If f is differentiable at $x \in \mathrm{E}$, then show that all partial derivatives $\mathrm{D}_i f_j(x)$ exists. Show also that converse of this implication is false. **8**

(b) Suppose $\mathrm{A} \in \mathrm{L}\left(\mathbf{R}^{n-m}, \mathbf{R}^n\right)$ and suppose $\mathrm{A}(h, 0) = 0$ implies $h = 0$ for each $h \in \mathbf{R}^n$. Show that for each $y \in \mathbf{R}^m$, the equation A(x, y) = 0 has one and only one solution of x in $\mathbf{R}^n$. **8**

6. (a) If f is real differentiable function in a connected open set $\mathrm{E} \subset \mathbf{R}^n$ and if $f'(x) = 0$ for each x in E, then show that f is constant in E. Can you drop connectedness of E ? Justify your answer. **8**

(b) Define $f(0, 0) = 0$ and $f(x, y) = \dfrac{x^3}{x^2 + y^2}$ if $(x, y) \neq (0, 0)$. Show that f is continuous in $\mathbf{R}^2$ and the restriction of f to any straighht line is differentiable. **8**

Section IV

7. (a) Suppose $\sum C_n$ converges. Let $f(x) = \sum_{n=0}^{\infty} C_n x^n$, $-1 < x < 1$. Show that $\lim_{x \to 1} f(x) = \sum_{n=0}^{\infty} C_n$. Hence or otherwise, show that if $\sum a_n$, $\sum b_n$ and $\sum C_n$ converge, where $C_n = \sum_{j=0}^{n} a_0 b_{n-j}$, then $\left(\sum a_n\right)\left(\sum b_n\right) = \sum C_n$. **8**

(b) If $\{\phi_n\}$ is orthonormal on $[a, b]$, if $f(x) \sim \sum_{n=1}^{\infty} C_n \phi_n(x)$, show that $\sum_{n=1}^{\infty} |C_n|^2 \leq \int_a^b |f(x)|^2 \, dx$. **8**

8. (a) Suppose K is a compact subset of $\mathbf{R}^n$ and $\{V_\alpha\}$ is an open cover of K. Show that there exists function $\psi_1, \psi_2, ..., \psi_s \in e\left(\mathbf{R}^n\right)$ such that :

$$\psi_1(x) + \psi_2(x) + ... + \psi_s(x) = 1$$

for every $x \in$ K. **8**

(b) Suppose T is a e'-mapping of an open set $E \subset \mathbf{R}^n$ into an open set $V \subset \mathbf{R}^m$, ϕ is a k-surface in E, and W is a k-form in V. Show that : **8**

$$\int_{T\phi} w = \int_{\phi} w_T$$

Compulsory Question

9. (a) Evaluate $\int_0^4 x\, d\left([x]-x\right)$, where $[x]$ denotes integral part of x.

(b) Suppose f is a bounded real function on $[a, b]$, and $f^3 \in \mathbf{R}$ on $[a, b]$. Does it follow that $f \in R$? Justify your answer.

(c) Show that a continuous map on a metric space need not have a fixed point.

(d) Show that a pointwise convergent sequence of functions need not be uniformly convergent.

(e) If $A \in L\left(\mathbf{R}^n, \mathbf{R}^m\right)$ and if $x \in \mathbf{R}^n$, show that $A'(x) = A$.

(f) If $x+y=u$, $y=uv$; find the Jacobian $\dfrac{\partial(x, y)}{\partial(u, v)}$.

(g) Find the interval of absolute convergence for the series $\sum_{n=1}^{\infty} \dfrac{x^n}{n^n}$.

(h) State (only) the Stirling's formula. **8×2=16**