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MDE/D-21 4314

ADVANCED ABSTRACT ALGEBRA

MM-401

Time : Three Hours] [Maximum Marks : 80

Note : Attempt *Five* questions in all, selecting *one* question from each Section. All questions carry equal marks.

Section I

1. (a) Let H be a proper subgroup of a finite p-group G. Prove that $H \neq N_G(H)$

(b) State and prove Zassenhauss's Lemma.

2. (a) State and prove Sylow second theorem.

(b) Let G be a group. Prove that :

$$o(G/Z(G)) \neq 77$$

Section II

3. (a) Let K/F and L/K be finite extensions. Prove that L/F is also finite and [L : F] = [L : K] [K : F].

(b) Find the degree of the splitting field of $(X^2 + X + 1)(X^3 - 2)$ Over Q.

4. (a) Prove that $GF(p^m)$ is a subfield of $GF(p^n)$ iff m divides n.

(b) Prove that a splitting field is a normal extension.

Section III

5. (a) Prove that an algebraic extension of a finite field is separable.

(b) Prove that if α and $\beta \in K/F$ are separable, then $F(\alpha, \beta)/F$ is a simple extension.

6. (a) State and prove Dede kind Lemma.

(b) Find the Calois group of the polynomial $X^4 - 8X^2 + 15$ over Q.

Section IV

7. (a) Let A and B be solvable normal subgroups of a group G such that both G/A and G/B are solvable. Prove that G is a solvable group.

(b) Let $H \Delta A_n$, $(n \geq 5)$. prove that if H contains one 3-cycle, then $H = A_n$.

8. (a) Prove that the polynomial $X^5 - 9X + 3$ is not solvable by radicals over Q.

(b) Prove that the angle $\pi/3$ cannot be trisected by using ruler and compass only.

Section V

9. (a) Prove that if G is non-Abelian group of order 125, than $o(Z(G) = 5$.

(b) Write down a composition series for the symmetric group S_4 of degree 4.

(c) Find the degree of $\sqrt{3}+1$ over Q.

(d) Prove that $Q\left(2^{\frac{1}{3}}\right)\Big|_{Q}$ is not normal.

(e) Find the Galois group of $\mathbb{C} \mid R$.

(f) Prove that field Q is perfect.

(g) Prove that a group of order 21 is solvable.

(h) Prove that the number $\sqrt{5}$ is constructible with ruler and compass only.