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MDE/J-21 4674

DIFFERENTIAL EQUATIONS–II

MM-411

Time : Three Hours] [Maximum Marks : 80

Note : Attempt *Five* questions in all, selecting *one* question from each Section. Q. No. **1** is compulsory.

1. (a) Explain the difference between positive definite and positive semi-definite functions. **2**

(b) Construct an example to show the orthogonality of eigen functions of a SLBVP. **2**

(c) Construct an example to illustrate Sturm fundamental comparison theorem. **2**

(d) Are saddle points of linear and non-linear systems always alike ? Explain with example. **2**

(e) Explain index of a critical point. **2**

(f) Give *two* examples, one for third order ODE and other for fourth order ODE, to illustrate principle of superposition. **2**

(g) Find the solution of the system corresponding to the differential equation :

$$a(x) = \frac{d^2y}{dx^2} + a'(x)\frac{dy}{dx} + b(x)y = 0$$

when solution of this differential equation is $y = f(x)$. **2**

(h) Show that every second order ODE can be written as a self-adjoint equation. **2**

Section I

2. (a) If u and v are two solutions of the differential equation $(p(t)y'(t))' + q(t)y(t) = 0$ on I. Let s be fixed on I, then prove that :

$w(t) = \frac{1}{c}\int_{t_0}^{t}[u(s)v(t) - u(t)v(s)]h(s)ds$, is the solution of $(p(t)y'(t))' + q(t)y(t) = h(t)$ such that $w(t_0) = w'(t_0) = 0$. **8**

(b) If $t_1, t_2, \ldots, t_n, \ldots$ are zeros of solution of $y''(t) + p(t)y'(t) + q(t)y(t) = 0$ on R^+, then prove that $t_n \to \infty$ as $n \to \infty$. **8**

3. (a) Obtain equivalent system of differential equations to a system $x'(t) = A(t)x(t)$ using Prüffer transformation. **8**

(b) Given a differential equation $y'' + p(x)y = 0$, where $p(x) > 0$ is continuous on $(0, \infty)$. Find the condition such that any solution of the given differential equation has infinite number of zeros on R^+. Prove your claim. **8**

Section II

4. (a) Prove that zeros of two real linearly independent solutions, of a second order linear differential equation, are separated. **8**

(b) Find the normal form of Bessel's equation whose one solution is $J_n(x)$. If $0 \le n < \frac{1}{2}$, then prove that every interval of length π contains at least *one* zero of $J_n(x)$ and if $n = \frac{1}{2}$, then prove that every zero of $J_n(x)$ is at a distance π from its successive zero. **8**

5. (a) Define a spiral point. Give an example of a system such that the point (2, 4) is a spiral point of that system. Illustrate directed description of its path. Find its stability. **8**

(b) If the roots of the characteristics equation of the system $\dot{x} = a_1 x + a_2 y$; $\dot{y} = b_1 x + b_2 y$ are purely imaginary, then prove that origin is a center. **8**

Section III

6. (a) Prove that for under-damped free oscillations of a coil spring, the critical point is spiral point while the critical point (0, 0) is a node in case of over-damped oscillations. Also prove that (0, 0) will be asymptotically stable in both the cases. Will the type of critical point, in both the cases, remain same when the system will become quasi-linear ? **8**

(b) Determine the stability of the critical point of :

$$\frac{dx}{dt} = (y+1)^2 - \cos x, \quad \frac{dy}{dt} = \sin(x+y)$$

by Liapunov's method. Determine the type of critical points of this non-linear system. **8**

7. (a) Define limit cycle. How can the non-existence of limit cycles of a system be established ? **8**

(b) Determine whether or not the differential equation :

$$\frac{d^2x}{dt^2} = \left(x^4 + x^2\right)\frac{dx}{dt} + \left(x^3 + x\right) = 0$$

has periodic solution. **8**

Section IV

8. (a) Given a Strum Liouville Boundary Value Problem :

$$\{(p(t)x'(t))\}' + q(t)x(t) + \lambda r(t)x(t) = 0$$

together with boundary conditions :

$m_1 x(a) + m_2 x'(a) = 0$, $m_3 x(b) + m_4 x'(b) = 0$;

where at least *one* of m_1 and m_2 and one of m_3 and m_4 are non-zero; a and b are finite real numbers; $p', q, r \in C[a, b]$; $r(t)$ is either positive or negative on $[a, b]$. Prove that all eigen values of this BVP are real. **8**

(b) What is a singular boundary value problem ? Give an example. Solve the boundary value problem :

$$\frac{\partial u}{\partial t}(x, t) - h^2 \frac{\partial^2 u}{\partial x^2}(x, t) = 0, \quad 0 < x < 1, \quad t > 0;$$

$u(x, 0) = u_0(x)$, $0 < x < 1$; $u(0, t) = u(1, t) = 0$, $t > 0$. **8**

9. (a) Show that the eigen values of the boundary value problem $x'' + \lambda x = 0$; $x(0) = 0$, $x(\pi) + x'(\pi) = 0$ satisfy the equation $\sqrt{\lambda} = -\tan\left(\sqrt{\lambda}\,\pi\right)$ and that the corresponding eigen functions are $\sin\left(\sqrt{\lambda_n}\, t\right)$. **8**

(b) Prove, through its construction, that Green's function is symmetric. **8**