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GSM/M-21 1614

MATHEMATICS

Special Functions and Integral Transforms

BM-242

Time : Three Hours] [Maximum Marks : 40

Note : Attempt *Five* questions in all, selecting *one* question from each Unit. Q. No. **1** is compulsory.

(Compulsory Question)

1. (a) Show that $J_{-\frac{1}{2}}(x) = J_{\frac{1}{2}}(x)\cot x$. **2**

(b) Prove that $\int_{-1}^{1} P_n(x)\left(1-2tx+t^2\right)^{-\frac{1}{2}} dx = \frac{2t^n}{2n+1}$. **2**

(c) Find the Laplace transform of $\int_0^1 \frac{\sin u}{u} du$. **2**

(d) If $\overline{f}(s)$ is the Fourier transform of $f(x)$, then show that $e^{-ias}\overline{f}(s)$ is the Fourier transforms of $f(x-a)$. **2**

Unit I

2. (a) Solve $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + \left(x^2 - 1\right)y = 0$ in series. **4**

(b) Prove that $\int x J_0^2(x)dx = \frac{x^2}{2}\left[J_0^2(x) + J_1^2(x)\right] + c$. **4**

3. (a) By the use of the substitution $y = \frac{u}{\sqrt{x}}$, show that the solution of the equation :

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + \left(x^2 - \frac{1}{4}\right)y = 0,$$

can be written in the form $y = c_1 \frac{\sin u}{\sqrt{x}} + c_2 \frac{\cos x}{\sqrt{x}}$. **4**

(b) Solve the equation $x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + \frac{1}{2}xy = 0$ in terms of Bessel's function. **4**

Unit II

4. (a) Prove that $\left(1 - 2xt + t^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} t^n P_n(x)$; $|x| \le 1$, $|t| < 1$. **4**

(b) Prove that :

$$\int_{-1}^{1}\left(1-x^2\right)P'_m(x)P'_n(x)dx=\begin{cases}0, & \text{if } m\neq n\\ \dfrac{2m(m+1)}{2m+1}, & \text{if } m=n\cdot\end{cases}$$

4

5. (a) Show that $H'_n(x)=2nH_{n-1}(x);\ n\geq 1$. **4**

(b) Prove that $H'_{2n}(0)$ and $H'_{2n+1}(0)=\dfrac{(-1)^n(2n+1)!}{n!}$.

4

Unit III

6. (a) Find the Laplace transform of $\sinh 3t\cos^2 t$. **4**

(b) Find the inverse Laplace transform of $\log\dfrac{s^2+1}{(s-1)^2}$.

4

7. (a) Convert the differential equation :

$$f''(t)-3f'(t)+2f(t)=4\sin t,$$

into an integral equation where $f(0)=1$, $f'(0)=-2$. **4**

(b) Solve the simultaneous equations :

$\frac{dx}{dt} = 5x + y$; $\frac{dy}{dt} = x + 5y$, when x (0) = –3, y (0) = 7 using Laplace transform method. **4**

Unit IV

8. (a) Find the Fourier cosine transform of $\frac{1}{1+x^2}$ and deduce the sine transform of $\frac{x}{1+x^2}$. **4**

(b) Find the inverse Fourier transform of $\overline{f}(s) = e^{-|s|y}$, where $y \in (-\infty, \infty)$. **4**

9. (a) Using Parseval's identity, prove that :

$$\int_0^\infty \frac{\sin ax}{x\left(a^2 + x^2\right)} dx = \frac{\pi}{2}\left(\frac{1 - e^{-a^2}}{a^2}\right).$$ **4**

(b) Using the Fourier sine transform, solve the differential equation $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ with the boundary condition (i) $u = u_0$ when $x = 0$, $t > 0$ and the initial condition (i) $u = 0$ when $t = 0$, $x > 0$. **4**