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MDQ/J-21 5510

ALGEBRAIC NUMBER THEORY

MM-509 (Option iii)

Time : Three Hours] [Maximum Marks : 80

Note : Attempt *Five* questions in all, selecting *one* question from each Section and the compulsory question.

Section I

1. Show that $\frac{\sqrt{2}}{3}$ is an algebraic number but not an algebraic integer. Find an integer in such that $m\frac{\sqrt{2}}{3}$ is an algebraic integer. **16**

2. Let $\phi_m(x)$ denote the mth cyclotomic polynomial, show that $\phi_m(x) \in \mathbb{Z}[x]$ and $\phi_m(x)$ is irreducible in $\mathbb{Q}[x]$ for every $m \geq 1$. **16**

Section II

3. Find integral basis for $\mathbb{Q}\left(\sqrt{2}, \sqrt{-3}\right)$ and for $\mathbb{Q}\left(\sqrt{5}, \sqrt{13}\right)$. **16**

4. Let $\mathbb{P}$ be a prime ideal of O_k. Then prove that $\mathbb{P}^{-1}$ is a fractional ideal and $\mathbb{P}\mathbb{P}^{-1} = O_k$. Also prove that any ideal of O_k can be written as a product of prime ideals uniquely. **16**

Section III

5. Compute the different D of $K = \mathbb{Q}\left(\sqrt{-2}\right)$ and of $K = \mathbb{Q}\left(\sqrt{-3}\right)$. **16**

6. Let $K > 0$ be a square free positive integer. Suppose that $K \equiv 1, 2 \pmod 4$, and K does not have the form $K = 3a^2 \pm 1$ for an integer a, consider the equation :

$$x^2 + K = y^3$$

Show that if 3 does not divide the class number of $\mathbb{Q}\left(\sqrt{-K}\right)$, then this equation has no integral solution. **16**

Section IV

7. State and prove law of quadratic reciprocity, using this find $\left(\frac{5}{p}\right)$. **16**

8. Show that $\mathbb{Q}\left(\sqrt{-31}\right)$ has class number 3. **16**

(Compulsory Question)

9. (a) If K is an algebraic number field and $\alpha \in K$, define trace of α. **2**

(b) If $K = \mathbb{Q}(i)$, find $\underset{K}{\mathrm{Tr}}(i)$. **2**

(c) Define fractional ideal. **2**

(d) Define index of an algebraic integer of degree n. **2**

(e) Define different of an algebraic number field. **2**

(f) State Minkowski's bound. **2**

(g) Find $\left(\frac{2}{3}\right)$. **2**

(h) Find $\left(\frac{3}{5}\right)$. **2**