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MDQ/J-21 **5507**

PARTIAL DIFFERENTIAL EQUATIONS
MM-508

Time : Three Hours] [Maximum Marks : 80

Note : Attempt *Five* questions in all, selecting at least *one* question from each Unit and the compulsory question. All questions carry equal marks.

(Compulsory Question)

1. (i) State Biouville's theorem.

(ii) From $u_t + b.Du = 0$, what physical insight you draw ?

(iii) If $x \in R^n - \{0\}$ write point dual to x with respect to $\partial B(0, 1)$.

(iv) State Duhamel's principle.

(v) Give physical interpretation of wave equation.

(vi) State Hamilton's principle.

(vii) Define Laplace transform and its inverse.

(viii) Explain the term weak solution.

Unit I

2. (a) Show that Laplace's equation is invariant under rotation.

(b) Derive mean value formulas for Laplace equation.

3. (a) Solve initial value problem :

$$u_t + b.Du + u = 0 \quad \text{in } R^n \times (0, \infty)$$

$$u = g \quad \text{on } R^n \times \{t = 0\}$$

where $b \in R^n$ is a constant.

(b) If u is harmonic, then prove that :

$$\left|D^{\alpha}u\left(x_0\right)\right| \leq \frac{C_k}{r^{n+k}}\|u\|L^1\left(B\left(x_0,\ r\right)\right)$$

for each ball $B(x_0, r) \subset U$ and each multi-index α of order $|\alpha| = k,$ where

$$C_0 = \frac{1}{\alpha(n)},\ C_K = \frac{\left(2^{n+1}nK\right)^K}{\alpha(n)}$$

$$K = 1, 2, ...$$

Unit II

4. (a) Obtain fundamental solution of heat equation.

(b) If u solves :

$$-\Delta u = f \quad \text{in U}$$

$$u = g \quad \text{on } \partial u,$$

then show that :

$$u(x) = -\int_{\partial U} g(y) \frac{\partial G}{\partial \nu}(x, y)\, dS(y) + \int_{U} f(y) G(x, y)\, dy$$

$x \in U$; in usual notations; $G(x, y)$ is Green's function.

5. (a) If $u \in A = \left\{ w \in C^2(\bar{U}) \mid w = g \text{ on } \partial U \right\}$, the partial differential equation $-\Delta u = f$ is equivalent to the statement that u minimizes energy :

$$I = \int_{U} \left(\frac{1}{2} |Dw|^2 - wf \right) dx,$$ establish the statement.

(b) Show that there exists only one solution $u \in C_{\perp}^2(\bar{U}_T)$ of :

$$u_t - \Delta u = f \quad \text{in } U_T$$
$$u = g \quad \text{on } \Gamma_T$$

Unit III

6. (a) For $u_{tt} - u_{xx} = 0$ in $R \times (0, \infty)$

$u = g,\ u_t = h$ on $R \times \{t = 0\}$

Derive d'Alembert's formula and interprete the solution.

(b) Prove that :

$$u(x, t, a, b) = a.x - tH(a) + b,\ a \in R^n, b \in R,$$

is complete integral of the Hamilton-Jacobi equation :

$$u_t + H(Du) = 0$$

7. (a) Solve using characteristics :

$$x_1 u_{x_1} + x_2 u_{x_2} = 2u,\ u(x_1,\ 1) = g(x_1)$$

(b) Obtain Kirchhoff's formula for the solution of the initial value problem :

$$u_{tt} - \Delta u = 0, \qquad \text{in } \mathrm{R}^n \times (0,\ \infty)$$

$$u = g,\ u_t = h \qquad \text{on } \mathrm{R}^n \times \{t = 0\}$$

Unit IV

8. (a) Solve Hamilton-Jacobi equation using the method of separation of variables.

(b) Discuss Lax-Oleinik formula.

9. (a) Define Fourier transform and its inverse, and show that :

$$\left(\mathrm{D}^{\alpha} u\right)^{\wedge} = (iy)^{\alpha}\, \hat{u} \text{ for each multi-index } \alpha.$$

(b) Get the Laplace transform of the :

$$v_t - \Delta v = 0 \qquad \text{in U} \times (0,\ \infty)$$

$$v = f \qquad \text{on U} \times \{t = 0\};$$

with respect to time.