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MDE/J-21 4673

COMPLEX ANALYSIS-II
MM-410

Time : Three Hours] [Maximum Marks : 80

Note : Attempt *Five* questions in all, selecting *one* question from each Section and the compulsory question. All questions carry equal marks.

Section I

1. (a) State and prove Montel's theorem.

(b) Prove that :

$$\sin \pi z = \pi z \prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)$$

by using Weierstrass Factorization theorem.

2. (a) Prove that :

$$\sqrt{\pi}\,\Gamma(2z) = 2^{2z-1}\Gamma(z)\,\Gamma\left(z+\frac{1}{2}\right)$$

(b) Show that :

$$\overline{|(z)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(z+n)} + \int_1^{\infty} e^{-t} t^{z-1}\, dt$$

fn $z \neq 0, -1, -2, \ldots\ldots$ (not for Re $z > 0$ alone)

Section II

3. (a) If Re $z > 1$, then show that :

$$\zeta(z) = \prod_{n=1}^{\infty} \left[\frac{1}{1 - p_n^{-z}} \right]$$

where $\{p_n\}$ is the sequence of prime numbers.

(b) State and prove Runge's theorem.

4. (a) State and prove the theorem of uniqueness of analytic continuation along a curve.

(b) Show that the function represented by the power series $f(z) = \sum_{n=0}^{\infty} z^{2^n}$ cannot be continued analytically.

Section III

5. (a) State and prove Monodromy theorem.

(b) If $f(z)$ is analytic in the closed disc $|z| \leq R$. Assume that $f(0) \neq 0$ and no zeros of $f(z)$ lies on $|z| = R$. If $z_1, z_2, \ldots, z_n$ are the zeros of $f(z)$ in the open disc $|z| < R$ each repeated as often as its multiplicity and $z = re^{i\theta}$, $0 < r, R, f(z) \neq 0$ then show that :

$$\log|f(z)| = -\sum_{i=1}^{n} \log\left|\frac{R^2 - \bar{z}_i z}{R(z - z_i)}\right| +$$

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)\log\left|f\left(Re^{i\phi}\right)\right|}{R^2 - 2Rr\cos(\theta - \phi) + r^2}\, d\phi$$

6. (a) State and prove Hadamard's three circle theorem.

(b) Show that if G is a bounded Dirichlet Region then for each $a \in G$ there is a Green's function on G with singularity at a.

Section IV

7. (a) If $f(z)$ is an entire function of order ρ and convergence exponent σ, then show that :

$$\sigma \leq \rho$$

(b) State and prove Bloch's theorem.

8. (a) State and prove Little-Picard theorem.

(b) State and prove Montel-Carathedory theorem.

Section V

(Compulsory Question)

9. (i) State Riemann mapping theorem.

(ii) Evaluate $\left\lceil\left(\frac{1}{2}\right)\right.$.

(iii) State Riemann's Hypothesis.

(iv) State Schwarz reflection principle.

(v) Define Poisson Kernel $P_r(\theta)$ and show that it periodic in θ with period 2π.

(vi) Define Dirichlet Region.

(vii) Find the order of $\cos z$.

(viii) Define univalent function.