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MDE/J-21 4671

MATHEMATICS
Real Analysis-II
MM-408

Time : Three Hours] [Maximum Marks : 80

Note : Attempt *Five* questions in all, selecting *one* question from each Section and the compulsory question.

Section I

1. (a) Let $\{A_n\}$ be a countable collection of sets of real numbers. Show that : **6**

$$m^*\left(\bigcup_{n=1}^{\infty} A_n\right) \le \sum_{n=1}^{\infty} m^*(A_n)$$

(b) Show that the class m of all measurable sets is a σ-algebra. **10**

2. (a) Construct a set which is not Lebesgue measurable. **8**

(b) Let $\{f_n\}$ be a sequence of measurable functions (with the same domain of definition). Show that the functions $\sup\{f_1, f_2, \ldots\ldots, f_n\}$, $\inf\{f_1, f_2, \ldots\ldots, f_n\}$, $\sup_n f_n$, $\inf_n f_n \overline{\lim} f_n$, $\underline{\lim} f_n$ are all measurable. **8**

Section II

3. (a) State and prove Egoroff's theorem. **10**

(b) Show that every sequence which is convergent in measure has an almost everywhere convergent subsequence. **6**

4. (a) Let f be defined and bounded on a measurale set E with mE finite. Show that f is measurable if and only if :

$$\inf_{f \leq \psi} \int_E \psi(x)dx = \sup_{f \geq \phi} \int_E \varphi(x)dx$$

for all simple functions φ and ψ. **10**

(b) If f and g are bounded measurable functions defined on a set E of finite measure, show that : **6**

$$\int_E (af + bg) = a\int_E f + b\int_E g.$$

Section III

5. (a) State and prove Monotone convergence theorem. Also show that this theorem need not hold for decreasing sequence of functions. **10**

(b) Let f be a non-negative function which is integrable over a set E. Show that for gives $\varepsilon > 0$ there is a $\delta > 0$ such that for every set $A \subset E$ with $mA < \delta$ where $\int_A f < \varepsilon$. **6**

6. State and prove Vitali's covering lemma. **16**

Section IV

7. (a) Let f be an integrable function on $[a, b]$, and suppose that :

$$F(x) = F(a) + \int_a^x f(t)dt.$$

Show that $F'(x) = f(x)$ for almost all x in $[a, b]$. **8**

(b) If φ is continuous function on (a, b) and if one derivative, of φ is nondecreasing, show thaet φ is convex. **8**

8. (a) If f and g are in L^p with $1 \le p \le \infty$, show that so is $f + g$ and

$$\|f+g\|_p \le \|f\|_p + \|g\|_p .$$

If $1 < p < \infty$, show that equality can hold only if there are non-negative constants α and β such that $\beta f = \alpha g$. **8**

(b) Show that the L^p spaces are complete. **8**

(Compulsory Question)

9. (a) Give an example to show that a set with outer measure zero need not be countable.

(b) Show that ϕ and $\mathbb{R}$ are measurable sets.

(c) Define almost uniform convergence.

(d) State (only) Lebesgue theorem regarding points of discontinuities of Riemann integrable functions.

(e) State (only) Lebesgue differentiation theorem.

(f) Give an example to show that strict inequality may occur in Fatou's lemma.

(g) State (only) Riesz representation theorem for bounded linear functionals or L^p spaces.

(h) State (only) Jensen's inequality. **8×2=16**