Roll No. .... **Total Pages : 04**

MDE/J-21 4670

ADVANCED ABSTRACT ALGEBRA-II

MM-407

Time : Three Hours] [Maximum Marks : 80

Note : Attempt *Five* questions in all, selecting *one* question from each Section and the compulsory question.

Section I

1. Let G be a group and let G′ be the derived group of G then prove that :

(i) $G' \Delta G$ **5**

(ii) $\frac{G}{G'}$ is abelian **5**

(iii) If $H \Delta G$, then $\frac{G}{H}$ is abelian if and only if $G' \subset H$. **6**

2. (a) Prove that a group of order p^n is nilpotent of class at most n. **8**

(b) Give an example of a group which is not nilpotent. **8**

Section II

3. (a) If $T \in A(V)$ has all its characteristic roots in F, then prove that there is a basis of V in which the matrix of T is triangular. **8**

(b) If V is n-dimensional over F and if $T \in A(V)$ has all its characteristic roots in F, then T satisfies a polynomial of degree n over F. **8**

4. (a) Prove that the elements S and T in $A_F(V)$ are similar in $A_F(V)$ if and only if they have the same elementary divisors. **8**

(b) If F is the field of rational numbers, find all possible rational canonical forms and elementary divisors for the 6×6 matrices in F_6 having $(x - 1)(x^2 + 1)^2$ as minimal polynomial. **8**

Section III

5. (a) Let A and B be R-submodules of R-modules M and N, respectively. Prove that : **8**

$$\frac{M \times N}{A \times B} \simeq \frac{M}{A} \times \frac{N}{B}$$

(b) Let R be a ring with unity. Prove that an R-module M is cyclic if and only if $M \simeq \frac{R}{I}$ for some left ideal I of R. **8**

6. (a) Let M be a finitely generated free module over a commutative ring R. Prove that all bases of M have the same number of elements. **8**

(b) Prove that every finitely generated module is a homomorphic image of a finitely generated free module. **8**

Section IV

7. (a) Determine whether $\mathbb{Z}$ as a $\mathbb{Z}$-module is artinian. **8**

(b) A subring of a noetherian ring need not be noetherian. Justify this statement. **8**

8. (a) State and prove Hilbert basis theorem. **8**

(b) Prove that a Boolean noetherian ring is finite and is indeed a finite direct product of fields with two elements. **6**

(Compulsory Question)

9. (a) Prove that $[a, b] = e$ if and only if $ab = ba$. **2**

(b) Prove that : **2**

$$[a, b]^g = \left[a^g, b^g\right]$$

(c) Define basic Jordan block belonging to λ. **2**

(d) Define index of nilpotence of a linear transformation. **2**

(e) Define finitely generated module. **2**

(f) Define simple module. **2**

(g) Let V be an n-dimensional vector space over a field F. Then V is artinian or not ? **2**

(h) If S is nonempty subset of a ring R, define left annihilator of S in R. **2**