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GSE/M-21 **1472**

MATHEMATICS

(Number Theory and Trigonometry)

Paper–BM-121

Time : Three Hours] [Maximum Marks : 40

Note : Attempt *five* questions in all, selecting *one* question from each section. Q. No. 1 is compulsory.

Compulsory Question

1. (a) Show that the difference between any number and its square is even. 2

(b) Evaluate $\phi(462)$. 2

(c) Prove that $\exp(2n\pi i) = 1$. 1

(d) Prove that $\cosh^2 x - \sinh^2 x = 1$. 1

(e) Find the principle and general values of $\log(-5)$. 2

SECTION–I

2. (a) Prove that an integer is divisible by 3 iff the sum of its digits is divisible by 3. 4

(b) Find the remainder on dividing the

$1! + 2! + 3!\ \ 4! + 5! + \ldots\ \ 100!$ by 12. 4

3. (a) Solve the congruence $222x \equiv 12 \pmod{18}$. 4

(b) If m is a prime number and a, b are two numbers less than m, then prove that

$a^{m-2} + a^{m-3}b + a^{m-4}b^2 + \ldots\ldots + b^{m-2}$

is a multiple of m. 4

SECTION–II

4. (a) Solve the congruences

$x \equiv 1 \pmod{4}$

$x \equiv 3 \pmod{5}$ and

$x \equiv 2 \pmod{7}$ simultaneously. 4

(b) Prove that $\phi(n) = \frac{n}{2}$ iff $n = 2^k$ for some integer $k \geq 1$.

4

5. (a) Find all n such that $d(n) = 10$. Hence find the least such value of n. 4

(b) Show that the smallest positive quadratic non-residue of an odd prime p is itself prime. 4

SECTION–III

6. (a) If $2 \cos \alpha = x + \frac{1}{x}$, $2 \cos \beta = y + \frac{1}{y}$; show that one of the values of $x^m y^n = \frac{1}{x^m y^n}$ is $2 \cos (m\alpha + n\beta)$. 4

(b) Solve $x^7 = 1$ and prove that the sum of the nth powers of the root is 7 or zero, according as n is or is not a multiple of 7. 4

7. (a) Show that $[\sin(\alpha - \theta) + e^{\pm i\alpha} \sin\theta]^n = \sin^{n-1}\alpha\, [\sin(\alpha - n\theta) + e^{\pm i\alpha} \sin n\theta]$. 4

(b) Form an equation whose roots are

$$\cos\frac{2\pi}{7}, \cos\frac{4\pi}{7} \text{ and } \cos\frac{8\pi}{7}.$$ 4

SECTION–IV

8. (a) If $i^{i^{i \ldots\ldots at \inf}} = A + iB$, principal values only being considered, prove that

(i) $\tan\dfrac{\pi A}{2} = \dfrac{B}{A}$

(ii) $A^2 + B^2 = e^{-\pi B}$. 4

(b) Separate $\tan^{-1}(x + iy)$ into real and imaginary parts. 4

9. (a) Show that $\dfrac{\pi}{2\sqrt{3}} = 1 - \dfrac{1}{3^2} + \dfrac{1}{5.3^2} - \dfrac{1}{7.3^3} + \cdots \infty$. 4

(b) Find the sum of the series :

$3 \sin\alpha + 5 \sin 2\alpha + 7 \sin 3\alpha + \ldots\ldots$ to n terms. 4