Roll No. .... Total Pages : 3

GSE/J-21 **1448**

NUMBER THEORY AND TRIGONOMETRY

Paper–BM-121

Time : Three Hours] [Maximum Marks : 27

Note : Attempt *five* questions in all, selecting *one* question from each section. Question No. 1 is compulsory.

Compulsory Question

1. (a) If a is odd, show that $a^2 \equiv 1 \pmod{8}$. 1

(b) Evaluate $\mu(270)$. 1

(c) Prove that $i^i = e^{-(4n+1)\frac{\pi}{2}}$. 1

(d) Solve the equation : $\tan^{-1} 2x + \tan^{-1} 3x = \frac{\pi}{4}$. 1

(e) If $z = x + iy$, show that $\sin^2 z + \cos^2 z = 1$. 1

SECTION–I

2. (a) Prove that the number of primes is infinite. 3

(b) Find the remainder obtained on dividing 3^{181} by 17. 2½

3. (a) Show that $x^{12} - y^{12}$ is divisible by 91, if x and y are co-prime to 91. 3

(b) If $(p - 1)! + 1 \equiv 0 \pmod{p}$, then show that p is a prime number. 2½

SECTION–II

4. (a) Find all integers that satisfy the congruences $x \equiv 1 \pmod{4}$, $x \equiv 0 \pmod{3}$, $x \equiv 5 \pmod{7}$ simultaneously. 3

(b) Show that $\phi(12^k) = 12^{k-1}\,\phi(12)$, where k is a positive integer. 2½

5. (a) Find all n such that $d(n) = 10$. Hence find the least such value of n. 3

(b) Evaluate $\left(-\frac{168}{11}\right)$. 2½

SECTION–III

6. (a) Show that the roots of the equation $(x - 1)^4 + x^4 = 0$ are given by $x = \frac{1}{2}\left[1 + i \cot \frac{2r+1}{8}\pi\right]$, $r = 0, 1, 2, 3.$ 3

(b) Prove that the four roots of the equation $16x^4 - 20x^2 + 5 = 0$ are $\pm\sin\frac{\pi}{5}$ and $\pm\sin\frac{2\pi}{5}$. 2½

7. (a) If $\tan(\theta + i\phi) = \sin(x + iy)$, prove that $\coth y . \sinh 2\phi = \cot x . \sin 2\theta$. 3

(b) If $\tan(\theta + i\phi) = \tan\alpha + i\sec\alpha$, show that

$$2\theta = n\pi + \frac{\pi}{2} = \alpha, \ e^{2\phi} = \pm\left(\cot\frac{\alpha}{2}\right).$$ 2½

SECTION–IV

8. (a) If the principal values are considered, prove that

$$\frac{(1+i)^{1-i}}{(1-i)^{1+i}} = \sin(\log 2) + i\cos(\log 2).$$ 3

(b) Solve the equation : $\cos^{-1} x + \sin^{-1}\frac{1}{\sqrt{5}} = \frac{\pi}{4}$. 2½

9. (a) Separate $\tanh^{-1}(x + iy)$ into real and imaginary parts. 3

(b) Find the sum of the series :

$$\sin\alpha + \frac{1}{2}\sin 2\alpha + \left(\frac{1}{2}\right)^2 \sin 3\alpha + \ldots\ldots \text{ to } \infty.$$ 2½