GSM/M-20

1614

SPECIAL FUNCTIONS AND INTEGRAL TRANSFORMS

Paper-BM-242

Time Allowed: 3 Hours] [Maximum Marks: 40

Note: Attempt **five** questions in all, selecting at least **one** question from each Unit. Question No. 1 is compulsory. All questions carry equal marks.

Compulsory Question

- 1. (a) Show that $\sin x = 2[J_1 J_3 + J_5 + \dots]$.
 - (b) Express the following in terms of Legendre's polynomials $4x^3 2x^2 3x + 8$.
 - (c) Evaluate $\int_0^\infty t e^{-2t} \cos t dt$.
 - (d) If the Fourier transform of f(x) is f(s), then the Fourier transform of f(ax) is $\frac{1}{a} \cdot \overline{f}\left(\frac{s}{a}\right)$.

UNIT-I

2. (a) Solve $(1-x^2)$ $\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 4y = 0$ in series about x = 0.

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(b) Prove that:

$$\frac{d}{dx} \Big[I_{_{n}}^{2}(x) + J_{_{n+1}}^{2}(x) \Big] = 2 \Bigg[\frac{n}{x} J_{_{n}}^{2}(x) - \frac{n+1}{x} J_{_{n+1}}^{2}(x) \Bigg]$$

and hence show that

$$J_0^2(x) + 2[J_1^2(x) + J_2^2(x) + J_3^2(x) + \dots] = 1.$$
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- 3. (a) Find the solution of $x \frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{1}{4}y = 0$ in terms of Bessel's function.
 - (b) Verify that the Bessel's function $J_{1/2}(x)=\sqrt{\frac{2}{\pi x}}$ sin x satisfies the Bessel's equation of order $\frac{1}{2}$.

UNIT-II

4. (a) Prove that:

$$P_{n}(x) = \frac{1}{n!2^{n}} \frac{d^{n}}{dx^{n}} (x^{2} - 1)^{n}.$$

(b) Prove that:

$$\int_{-1}^{1} x^{2} P_{n+1}(x) P_{n-1}(x) dx = \frac{2n(n+1)}{(2n-1)(2n+1)(2n+3)}$$

Hence deduce that:

$$\int_0^1 x^2 P_{n+1}(x) P_{n-1}(x) dx = \frac{n(n+1)}{(2n+3)(4n^2-1)}.$$

5. (a) Show that :

$$H_{n}(x) = 2^{n} \left[exp \cdot \left(-\frac{1}{4} \frac{d^{2}}{dx^{2}} \right) \right] x^{n}.$$

(b) Prove that:

$$\int_{-\infty}^{\infty} x^{2} e^{-x^{2}} \left[H_{n}(x) \right]^{2} dx = \sqrt{\pi} 2^{n} n! \left(n + \frac{1}{2} \right).$$
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UNIT-III

- 6. (a) Find the Laplace transform of function sinh³ 2t.
 - (b) Find:

$$L^{-1} \left\lceil \frac{1}{s(s-6)^4} \right\rceil$$
.

7. (a) Solve:

$$\int_0^t \frac{f(u)}{\sqrt{t-u}} = 1 + 2t - t^2.$$

(b) Solve the following equation by transform method:

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = e^{-t}$$
; where $y(0) = y'(0) = 1$. 4

UNIT-IV

- 8. (a) Find the Fourier cosine transform of e^{-x^2} . 4
 - (b) Find the Fourier transform of $f(x)=\begin{cases} 1 & \text{for } |x|<1\\ 0 & \text{for } |x|>1 \end{cases}$ Hence find $\int_0^\infty \frac{\sin x}{x} dx$.
- 9. (a) The initial temperature of an infinite bar is given $by \ \theta(x) = \begin{cases} \theta_0 & \text{for } |x| < a \\ 0 & \text{for } |x| > a \end{cases} \text{ determine the temperature }$ at any point x and at any instant t.
 - (b) Using Parseval's identity prove that:

$$\int_0^\infty \frac{x^2 dx}{(x^2+1)^2} = \frac{\pi}{4} .$$