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GSE/M-20 **1474**

MATHEMATICS

(Vector Calculus)

Paper : BM-123

Time : Three Hours] [Maximum Marks : 40

Note : Attempt *five* questions in all. Question No. 1 is compulsory. Select *one* question from each section.

Compulsory Question

1. (a) Find the volume of cuboid whose coterminous edges are $11\hat{i}$, $2\hat{j}$, $13\hat{k}$. 1

(b) Find the unit tangent vector to any point on the curve $x = a \cos t$, $y = a \sin t$, $z = bt$. 1

(c) If $\vec{f}$ and $\vec{g}$ are irrotational, prove that $\vec{f} \times \vec{g}$ is solenoidal. 2

(d) Define Orthogonal curvilinear co-ordinates. 2

(e) Show that $\oint \vec{r} \cdot d\vec{r} = 0$. 2

SECTION–I

2. (a) Show that $[\vec{a}+\vec{b},\ \vec{b}+\vec{c},\ \vec{c}+\vec{a}] = 2[\vec{a},\ \vec{b},\ \vec{c}]$. 4

(b) If $\vec{a}, \vec{b}, \vec{c}$ are three vectors such that $\vec{a} \times \vec{b} = \vec{c}$ and $\vec{b} \times \vec{c} = \vec{a}$, show that three vectors $\vec{a}, \vec{b}, \vec{c}$ are orthogonal in pairs and $|\vec{b}| = 1$, $|\vec{c}| = |\vec{a}|$. 4

3. (a) A particle moves along the curves, $x = 3t^2$, $y = t^2 - 2t$, $z = t^3$. Find its velocity and acceleration at $t = 1$ in the direction of vector $\vec{a} = \hat{i} + \hat{j} - \hat{k}$. 4

(b) Prove that the necessary and sufficient condition for the vector function $\vec{f}$ of a scalar variable t to have a constant magnitude is $\vec{f} \cdot \frac{d\vec{f}}{dt} = 0$. 4

SECTION–II

4. (a) Prove that $\nabla^2(\vec{r} \cdot \vec{r}) = \left(\frac{4}{r}\right)\vec{r}$, where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $r = |\vec{r}|$. 4

(b) Find the directional derivative of the function $\phi(x, y) = \frac{xy}{x^2 + y^2}$ at the point (0, 1) along a line making an angle of 30° with +ve direction of x-axis. 4

5. (a) Evaluate div. $\left(\frac{\vec{r}}{r}\right)$, where $\vec{r} = x\vec{i} + y\hat{j} + z\hat{k}$ and $|\vec{r}| = r$. 4

(b) Show that the function $\frac{1}{r}$, where

$$r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

is harmonic function, provided $r \neq 0$. 4

SECTION–III

6. (a) Show that in orthogonal coordinates

$$\nabla \cdot (\vec{A}_1 \hat{e}_1) = \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial u} (A_1 h_2 h_3). \quad 4$$

(b) Prove that cylindrical co-ordinates system is self-reciprocal. 4

7. (a) Prove that $\frac{d}{dt}(\hat{e}_\phi) = -\sin\theta \frac{d\phi}{dt} \hat{e}_r - \cos\theta \frac{d\phi}{dt} \hat{e}_\theta$. 4

(b) Transform the following function from spherical to cartesian system :

$$\vec{f} = 3ar^2 \sin\theta \cos\phi \, \hat{e}_r + 2a^2 r \cos\theta \sin\phi \, \hat{e}_\theta + r^3 \hat{e}_\phi. \quad 4$$

SECTION–IV

8. (a) State and prove Green's theorem. 4

(b) Find the circulation of $\vec{f}$ around the curve C, where $\vec{f} = y\hat{i} + z\hat{j} + x\hat{k}$ and C is the circle $x^2 + y^2 = 1$, $z = 0$. 4

9. (a) Evaluate $\iint_S \vec{f} \cdot \hat{n} \, dS$, where $\vec{f} = 18z\hat{i} - 12\hat{j} + 3y\hat{k}$ and S is the surface of the plane $2x + 3y + 6z = 12$ which lies in the first octant. 4

(b) If $\vec{r} = 5t^2\hat{i} + t\hat{j} - t^3\hat{k}$, prove that

$$\int_1^2 \left[\vec{r} \times \frac{d^2\vec{r}}{dt^2}\right] dt = -14\hat{i} = 75\hat{j} - 15\hat{k}.$$ 4