Roll No. .... Total Pages : 3

GSE/M-20 **1472**

MATHEMATICS
(Number Theory and Trigonometry)
Paper : BM-121

Time : Three Hours] [Maximum Marks : 40

Note : Attempt *five* questions in all. Question No. 1 is compulsory. Select *one* question from each section.

Compulsory Question

1. (a) If $a|bc$ and $(a, b) = 1$, then prove that $a|c$. 2

(b) Find the least positive integer (mod 11) to which 282 is congruent. 2

(c) Find all possible values of n which satisfies $\phi(n) = 23$. 2

(d) Express $\cos^6 \theta$ in terms of cosines of multiples of θ. 1

(e) Prove that $\tan^{-1}\left(\frac{x}{\sqrt{a^2 - x^2}}\right) = \sin^{-1}\left(\frac{x}{a}\right)$. 1

SECTION–I

2. (a) Prove that there are infinitely many pairs of integers x, y satisfying $x + y = 100$ and $(x, y) = 10$ simultaneously. 4

(b) Solve the congruence $342x \equiv 5 \pmod{13}$. 4

3. (a) State and prove Wilson's theorem. 4

(b) Show that $n^{16} - a^{16}$ is divisible by 85 if n and a are coprime to 85. 4

SECTION–II

4. (a) Find all integers that give the remainder 1, 3, 2 when divided by 4, 5, 7 respectively. 4

(b) Prove that $\phi(n) = \phi(n + 2)$ is satisfied by $n = 2(2p - 1)$ whenever p and $2p - 1$ are both odd prime. 4

5. (a) Find the highest power of 180 in 102!. 4

(b) Evaluate $\left(-\frac{168}{11}\right)$. 4

SECTION–III

6. (a) If $(1 + x)^n = p_0 + p_1 x + p_2 x^2 + \ldots$, show that

(i) $p_1 - p_3 + p_5 + \ldots\ldots = 2^{n/2} \sin \frac{n\pi}{4}$.

(ii) $p_0 - p_2 + p_4 + \ldots\ldots = 2^{n/2} \cos \frac{n\pi}{4}$. 4

(b) Show that

$$\tan\frac{\theta}{7} + \tan\frac{\theta + \pi}{7} + \cdots + \tan\frac{\theta + 6\pi}{7} = 7 \tan \theta.$$ 4

7. (a) Express $\sin^6\theta\cos^2\theta$ in a series of cosines of multiples of θ. 4

(b) Separate $\tanh(x+iy)$ into real and imaginary parts. 4

SECTION–IV

8. (a) Prove that principal value of $\dfrac{(a+ib)^{p+iq}}{(a-ib)^{p-iq}}$ is

$$\cos 2(p\alpha + q\log r) + i\sin 2(p\alpha + q\log r),$$

where $r=\sqrt{a^2+b^2}$ and $\alpha = \tan^{-1}\dfrac{b}{a}$. 4

(b) Prove that

$$\sin^{-1}(\operatorname{cosec}\theta) = [2n+(-1)^n]\frac{\pi}{2} + i(-1)^n\log\cot\frac{\theta}{2}.$$ 4

9. (a) Show that

$$\frac{\pi}{4} = \frac{17}{21} - \frac{713}{81.343} + \cdots + \frac{(-1)^{n+1}}{2n-1}\left[\frac{2}{3}\cdot 9^{1-n} + 7^{1-2n}\right] + \cdots$$

4

(b) Sum the series

$$\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{13} + \cdots + \text{ to } n \text{ terms}$$

and deduce the sum to infinity. 4