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MDQ/M-20 5510

ALGEBRAIC NUMBER THEORY

MM-509 (Opt. iii)

Time : Three Hours] [Maximum Marks : 80

Note : Attempt *Five* questions in all, selecting *one* question from each Section and the compulsory question.

Section I

1. Show that Liouville's theorem holds for α, where α is real or complex algebraic number of degree greater than one. **16**

2. Let m be any positive integer and n be any integer ≥ 3. Show that $x^n + y^2 = m$ has only finitely many integral solutions. **16**

Section II

3. Let K/**Q** be an algebraic number field of degree n. Show that $d_k \in \mathbf{Z}$ and $d_k \equiv 0$ or 1 (mod 4). **16**

4. If $\boldsymbol{a}$ and $\boldsymbol{b}$ are ideals of O_k. Let $\boldsymbol{a} = \prod_{i=1}^{r} \boldsymbol{p}_i^{e_i}$, $\boldsymbol{b} = \prod_{i=1}^{r} \boldsymbol{p}_i^{f_i}$ with $e_i, f_i \in \mathbf{Z}$ and $e_i, f_i \geq 0$. Show that :

$$\text{g.c.d.}(\boldsymbol{a}, \boldsymbol{b}) = \prod_{i=1}^{r} \boldsymbol{p}_i^{\min(e_i, f_i)}$$ and

$$\text{L.C.M.}(\boldsymbol{a}, \boldsymbol{b}) = \prod_{i=1}^{r} \boldsymbol{p}_i^{\max(e_i, f_i)}$$

16

Section III

5. Find a prime ideal factorization of (2), (5) and (11) in $\mathbf{Z}[i]$. **16**

6. Show that the equation $x^2 + 5 = y^3$ has no integral solution. **16**

Section IV

7. Let S be Gauss sum. Prove that :

$$s^2 = \left(-\frac{1}{p}\right)^p \quad \text{and} \quad s^2 \equiv \left(\frac{q}{p}\right) s \pmod q$$

where p and q are distinct odd primes. **16**

8. Show that $\mathbf{Q}\left(\sqrt{-11}\right)$ has class number 1 and $\mathbf{Q}\left(\sqrt{-15}\right)$ has class number 2. **16**

Section V

(Compulsory Question)

9. (a) If k is an algebraic number field and $\alpha \in k$, define norm of α. **2**

(b) If $k = \mathbf{Q}(i)$, find $N_k(i)$. **2**

(c) Define Dedekind domain. **2**

(d) Define index of submodule in module. **2**

(e) Define ramification degree of prime ideal $\boldsymbol{p}_i$. **2**

(f) Define Hurwitz constant H_k. **2**

(g) Find $\left(\frac{2}{5}\right)$. **2**

(h) Find $\left(\frac{5}{7}\right)$. **2**