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# MDQ/M-20 5507

# PARTIAL DIFFERENTIAL EQUATIONS

## MM-508

Time : Three Hours] [Maximum Marks : 80

**Note :** Attempt *Five* questions in all, selecting *one* question from each Section and the compulsory question. All questions carry equal marks.

### Compulsory Question

**1.** (i) Give an example of quasilinear partial differential equations with explanation.

(ii) State properties of harmonic functions.

(iii) Write and explain mean value property of heat equation.

(iv) Write properties of Green's function.

(v) In $u_{tt} - u_{xx} = 0$, change the variables $\xi = x+t, \eta = x-t$ and get the resultant equation.

(vi) What is reflection method ?

(vii) Explain jump conditions along shocks.

(viii) Define Fourier transform and its inverse.

## Section I

**2.** (a) If $v$ is subharmonic, then prove that :

$$v(x) \leq \fint_{B(x, r)} v \, dy, \text{ for all } B(x, r) \subset U$$

(b) Solve :

$$u_t + b.Du + Cu = 0 \text{ in } R^n \times (0, \infty)$$

$$u = g \text{ on } R^n \times \{t = 0\}$$

where $C \in R$ and $b \in R^n$ are constant.

**3.** (a) Use Poisson's formula for ball to prove :

$$r^{n-2} \frac{r - |x|}{(r + |x|)^{n-1}} u(0) \leq u(x) \leq r^{n-2} \frac{r + |x|}{(r - |x|)^{n-1}} u(0)$$

where $u$ is positive and harmonic in $B^{\circ}(0, r)$

(b) State and prove Liouville's theorem.

## Section II

**4.** (a) Prove that Green's function in symmetric.

(b) Derive Green's function for the unit ball.

**5.** Derive mean value formula for Heat equation.

## Section III

**6.** (a) By changing the variables $\xi = x + t, \eta = x - t$, show that $u_{tt} - u_{xx} = 0$ if and only if $u_{\xi\eta} = 0$, and hence get the general solution of wave equation; in the form :

$$u = F(x+t) + G(x-t)$$

(b) Show that there exists at most one function $u \in C^2(\bar{U}_T)$ that solves :

$$u_{tt} - \Delta u = f \text{ in } U_T$$

$$u = g \text{ on } \Gamma_T$$

$$u_t = h \text{ on } U \times \{t = 0\}$$

**7.** (a) Solve :

$$u_{x_1} + u_{x_2} = u^2 \text{ in } U$$

$$u = g \text{ on } \Gamma, \ U = \{x_2 > 0\}$$

$$\Gamma = \{x_2 = 0\}$$

using characteristics.

(b) Write full note on Legendre transform.

## Section IV

**8.** Assume H is $C^2$ and satisfies H is convex and $\lim_{|p|\to\infty} \frac{H(p)}{|p|} = +\infty$. Also $g : R^n \to R$ is Lipschitz continuous i.e. $\text{lip}(g) = \sup_{x,\, y \in R^n} \left\{ \frac{|g(x) - g(y)|}{|x - y|} \right\} < \infty$, then prove the uniqueness of weak solution of :

$$H_t + H(Du) = 0 \quad \text{in } R^n \times (0, \infty)$$
$$u = g \quad \text{on } R^n \times \{t = 0\}$$

**9.** (a) Find fundamental solution of heat equation by using Fourier transform.

(b) Solve :

$$u_{tt} - \Delta u = 0 \text{ in } R^n \times (0, \infty)$$
$$u = g,\ u_t = 0 \text{ on } R^n \times \{t = 0\}$$

where $n$ is odd, $g$ is smooth with compact support using Laplace Transform.