Roll No. .... **Total Pages : 04**

MDE/M-20 **4665**

ADVANCED ABSTRACT ALGEBRA-II

MM-407

Time : Three Hours] [Maximum Marks : 80

Note : Attempt *Five* questions in all, selecting *one* question from each Section and the compulsory question.

Section I

1. (a) Find elements of S_3 which are commutators. **8**

(b) State and prove three subgroup lemma of P. Hall. **8**

2. Let G be a nilpotent group of class C. Then prove that :

(a) Every factor group of G is nilpotent of class $\leq$ C. **8**

(b) Every subgroup of G is nilpotent of class $\leq$ C. **8**

Section II

3. Prove that two nilpotent linear transformations are similar if and only if they have the same invariants. **16**

4. (a) Let T $\in A_F(V)$ have all its distinct characteristic roots, $\lambda_1, \lambda_2, \ldots\ldots, \lambda_k$, in F. Prove that a basis of V can be found in which the matrix of T is of the form :

$$\begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_k \end{pmatrix}$$

Where each

$$J_i = \begin{pmatrix} B_{i_1} & & & \\ & B_{i_2} & & \\ & & \ddots & \\ & & & B_{ir_i} \end{pmatrix}$$

and where $B_{i_1}, B_{i_2}, \cdots B_{ir_i}$ are basic Jordan blocks belonging to λ_i. **10**

(b) Prove that the matrix :

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 1 & 1 & 0 \end{pmatrix}$$

is nilpotent and find its invariants and Jordan form. **6**

Section III

5. (a) Let R be a ring with unity, and let M be a R-Module, prove that the following statements are equivalent :

(i) M is simple

(ii) $M \neq (0)$, and M is generated by any $0 \neq x \in M$.

(iii) $M \simeq \frac{R}{I}$, where I is a maximal left ideal of R. **10**

(b) State and prove Schur's lemma. **6**

6. (a) Let M be a free R-Module with a basis $\{e_1, e_2, ..., e_n\}$. Prove that $M \simeq R^n$. **8**

(b) Let V be a non-zero finitely generated vector space over a field F. Prove that V admits a finite basis. **8**

Section IV

7. (a) Determine whether $\mathbb{Z}$ as a $\mathbb{Z}$-module is noetherian. **8**

(b) A subring of an artinian ring need not be artinian. Justify this statement. **8**

8. State and prove Wedderburn-Artin theorem. **16**

(Compulsory Question)

9. (a) Prove that : **2**

$$[a, b]^{-1} = [b, a]$$

(b) Prove that : **2**

$$a^b = a[a, b]$$

(c) Define Companian matrix of : **2**

$$f(x) = \lambda_0 + \lambda_1 x + ... + \lambda_{r-1} x^{r-1} + x^r .$$

(d) Define characteristic polynomial of a linear transforamtion. **2**

(e) Define cyclic module. **2**

(f) Define completely reducible module. **2**

(g) Define finitely cogenerated R-module. **2**

(h) Let V an *n*-dimensional vector space over a field F. Then V is noetherian or not ? **2**