Section - IV

Throughout this chapter X denotes the locally compact Hausdroff Topological Space

R.K:- We know that the locally compact Hausdroff space is completely regular

Defn: A real valued function f defined on X is said to have compact support if $\exists$ a compact set C in X such that $f = 0$ on $X - C$

Notation: We shall write $\mathcal{L}(X)$ or brief $\mathcal{L}$, for a class of all cts. real valued functions on X with compact support

i.e $\mathcal{L}(X) = \mathcal{L} = \{ f : f : X \to \mathbb{R}$ is cts & f has compact support $\}$

Thm 1:- If f & g belong to $\mathcal{L}$ i.e $f, g \in \mathcal{L}$ & c is a real number then the following functions belongs to $\mathcal{L}$:
$f+g$, cf, fg, $|f|$, $f \cup g$, $f \cap g$, f^+, f^-

Proof:- let $f, g \in \mathcal{L}$

$\Rightarrow$ $\exists$ compact sets C & D such that $f(x) = 0$ on $X - C$ & $g(x) = 0$ on $X - D$

Now $C \cup D = E$ is a compact set

Now $x \notin E \Rightarrow x \notin C$ and $x \notin D$

$\Rightarrow f(x) = 0$ & $g(x) = 0$

$\Rightarrow f(x) + g(x) = 0$ on $X - E$

$\Rightarrow (f+g)(x) = 0$ on $X - E$

$\Rightarrow f+g = 0$ on $X - (C \cup D)$

Also $f+g$ is a real valued continuous function

$\therefore f+g \in \mathcal{L}$

(ii) If $c = 0$ then the result holds

If $c \neq 0$

Since $f \in \mathcal{L}$ $\therefore \exists$ a compact set D such that $f(x) = 0$ on $X-D$

& $f(x)$ is cts.

$\Rightarrow c f(x)$ is cts (∵ c is a real no.)

$f(x) = 0$ on $X-D$

$c f(x) = 0$ on $X-D$

$\therefore cf = 0$ on $X-D$

& $cf(x)$ is cts

$\therefore cf \in \mathcal{L}$

(iii) Since $f, g \in \mathcal{L}$

$\therefore \exists$ compacts sets C & D such that

$f(x) = 0$ on $X-C$

& $g(x) = 0$ on $X-D$

let $E = C \cup D$ then E is a compact set

$x \notin E$ ~~$f(x)\, g(x) = 0$~~

$\Rightarrow x \notin C$ & $x \notin D$

$\Rightarrow f(x) = 0$ & $g(x) = 0$

$\Rightarrow f(x)\, g(x) = 0$

$fg(x) = 0$ on $X-E$

$\Rightarrow fg = 0$ on $X - C \cup D$

Also $fg(x)$ is cts

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$\therefore$ $fg \in \mathcal{L}$

(iv) Since $f \in \mathcal{L}$

$\Rightarrow \exists$ a compact set C such that

$f(x) = 0$ on $X - C$

& $f(x)$ is cts

$\Rightarrow |f(x)|$ is also cts.

Now $f(x) = 0$ on $X - C$

$|f(x)| = 0$ on $X - C$

$|f|(x) = 0$ on $X - C$

$\Rightarrow |f| = 0$ on $X - C$

$\therefore |f| \in \mathcal{L}$

we know that

(v) $f \vee g = \max\{f(x), g(x)\}$

$= \dfrac{(f+g) + |f-g|}{2}$

Since $f, g \in \mathcal{L}$

$\Rightarrow f+g \in \mathcal{L}$ (by (i))

$g \in \mathcal{L}$.

$\Rightarrow -g \in \mathcal{L}$

$\Rightarrow f-g \in \mathcal{L}$

$\Rightarrow |f-g| \in \mathcal{L}$ (By (iv))

$\therefore \dfrac{(f+g) + |f-g|}{2} \in \mathcal{L}$ (By part (ii) here $c = \frac{1}{2}$)

$\Rightarrow f \vee g \in \mathcal{L}$

Illy $f \cap g \in \mathcal{L}$

$\because f \cap g = \frac{(f+g) - |f-g|}{2}$

(vi) $f^+ = \frac{f + |f|}{2}$

Since $f \in \mathcal{L}$

$\Rightarrow |f| \in \mathcal{L}$ (By (iv))

$\Rightarrow f + |f| \in \mathcal{L}$ (By (i))

$\Rightarrow \frac{f+|f|}{2} \in \mathcal{L}$ (By iii here $c = \frac{1}{2}$)

$\Rightarrow f^+ \in \mathcal{L}$

Similarly, $f^- \in \mathcal{L}$

Thm:- If $x \in X$ & V is nhd of x, $\exists$ an f in $\mathcal{L}$ such that $f(x) = 1$, $f = 0$ on $X - V$ & $0 \leq f \leq 1$

Proof:- Now V is nhd of x
& X is locally compact
$\therefore \exists$ a nhd U of x such that
$\bar{U}$ is compact & $\bar{U} \subset V$.

Now, Since X is locally compact & hausdorff space $\therefore X$ is completely regular & U is nhd of x $\therefore \exists$ a real valued & cts function f on X s.t $f(x) = 1$, $f = 0$ on $X - U$
& $0 \leq f \leq 1$

$\Rightarrow f = 0$ on $X - U$

$\Rightarrow f = 0$ on $X - \bar{U}$ $\left(\because U \subset \bar{U} \subset V \atop \therefore X - V \subset X - \bar{U} \subset X - U \right)$

$\Rightarrow f = 0$ on $X - V$

Page :
Date :

$\therefore f \in \mathcal{L}$ $\quad f(x) = 1 \quad f = 0$ on $X - V$

& $0 \leq f \leq 1$

Hence proved