Order of an integral function:→ Let $f(z)$ be any integral f^n, then f is of finite order, if $\exists$ a constant k and $r_0 > 0$ such that

$$|f(z)| < \exp(|z|^K) \quad \text{for } |z| = r_0 \to (1)$$

The infimum of all the values of K for which eqn. (1) holds, is called the order of $f(z)$, i.e,

$$\rho = \inf\{K ; |f(z)| < \exp(|z|^K)\}$$

for sufficiently large r_0, is known as order of $f(z)$.

Note⇒ If $\exists$ no such K, then the integral f^n $f(z)$ is said to be of infinite order.

Exponent of Convergence:→ Let $\langle z_n \rangle$ be an arbitrary seq. of non-zero complex numbers such that

$$|z_1| \le |z_2| \le \cdots < |z_n| \to \infty \text{ as } n \to$$

Consider the series,

$$\sum_{n=1}^{\infty} \frac{1}{|z_n|^{\alpha}}, \quad \alpha \geq 0 \ (\alpha \in \mathbb{R}) \longrightarrow (1)$$

If this series converges for some value of α_0, then it will converge $\forall$ $\alpha > \alpha_0$ (By p series test)

Now, the greatest Lower Bound of all the values of α for which eqn. (1) converges ~~for which eqn (1)~~ is called exponent of convergence of the sequence $\langle z_n \rangle$.

Clearly, exponent of convergence is a non-negative number. Let it be denoted by σ.

then,

Exponential of convergence is,

$$\sigma = \inf \left\{ \alpha > 0 ; \sum_{n=1}^{\infty} \frac{1}{|z_n|^{\alpha}} < \infty \right\}$$

Note ⇒ (1) If the series (1) diverges for every $\alpha > 0$, then we say that $\boxed{\sigma = \infty}$.

(2) If $\sigma = 0$, then there exists finite

no. of zeros of $f(z)$.

③ If $\sigma > 0$, then there exists infinite no. of zeros of $f(z)$.

<u>Borel's Thm</u> :→ The order of a canonical product is equal to the exponent of convergence of its zeros.

<u>Proof</u> ⇒ Let ρ and σ be the order and exponent of convergence of the canonical product (respectively).

<u>Claim</u> ⇒ $\sigma = \rho$

For this, we will prove ① $\sigma \leq \rho$
and ② $\rho \leq \sigma$

<u>Claim①</u> ⇒ <u>To prove</u> ⇒ $\boxed{\sigma \leq \rho}$

i) If ρ is infinite, then $\sigma \leq \rho$ is trivially true.

ii) If no. of zeros of $f(z)$ is finite, then $\sigma = 0$ and thus,
$\sigma \leq \rho$ holds ($\because \rho \geq 0$)

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$f(z)$ is taken to be an entire f^n.

(iii) let us suppose that ρ is finite and there are infinitely many zeros of $f(z)$ which are arranged in the form of a seq. $\{z_n\}$ s.t

$$|z_1| \le |z_2| \le |z_3| \le \cdots$$

and $|z_n| \to \infty$ as $n \to \infty$

Now, By def^n of $N(r)$, it can be easily observed that

$$N(|z_n|) \ge n \longrightarrow ①$$

The strict inequality will hold, when $|z_n| = |z_{n+1}|$

Notation of $N(r)$ meaning :→

$N(r)$ = No. of zeros of an entire f^n $f(z)$ in the closed disc $|z| = r$

Also, we know that,

$$N(r) \le r^{\rho+\epsilon} \text{ for larger.}$$

[∵ If $f(z)$ is an entire f^n of order ρ, then for $\epsilon > 0$, $N(r) \le r^{\rho+\epsilon}$ holds for sufficiently large r]

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Thus,

$$N(|z_n|) \le |z_n|^{\rho+\epsilon} \longrightarrow ②$$

For sufficiently large value of n and $\epsilon > 0$, by ① and ② we can write

$$|z_n|^{\rho+\epsilon} > n \longrightarrow ③$$

since, $|z_n| \to \infty$ as $n \to \infty$

so, we may assume that $|z_n| \ge 1$.

Let us suppose, there exists a 't' for which $t > \rho$.

for $\epsilon > 0$, $\rho + \epsilon > 0$

$$\Rightarrow \quad t > \rho > \rho + \epsilon$$

$$\Rightarrow \boxed{\frac{t}{\rho+\epsilon} > 1}$$

Raising Power as $\frac{t}{\rho+\epsilon}$ in eqn. ③,

$$|z_n|^t \ge n^{t/\rho+\epsilon}$$

Consequently,

$$\sum_{n=1}^{\infty} |z_n|^{-t} \le \sum_{n=1}^{\infty} n^{-t/\rho+\epsilon} \rightarrow ④$$

or, $\sum_{n=1}^{\infty} \frac{1}{|z_n|^t} \leq \sum_{n=1}^{\infty} \frac{1}{n^{t/\rho+\epsilon}}$ → (4')

since, $\frac{t}{\rho+\epsilon} > 1$

⇒ series on RHS of (4) is cgt (By p series test)

⇒ $\sum_{n=1}^{\infty} \frac{1}{|z_n|^t}$ is also convergent

for every $t > \rho$. → (5)

(By comparison test)

But by the def^n of exponent of convergence, we have

$$\sigma = \inf \left\{ t > 0 ; \sum_{n=1}^{\infty} |z_n|^{-t} < \infty \right\}$$

Since, eqn. (5) holds for every $t > \rho$

⇒ $\boxed{\sigma \leq \rho}$

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Claim ② $\Rightarrow$ To prove $\Rightarrow$ $\boxed{\rho \leq \sigma}$

Proof $\Rightarrow$ By weirstrass factorisation thm, we know that

$$E_k(w) = (1-w)\exp\left(w + \frac{w^2}{2} + \cdots + \frac{w^k}{k}\right) ; k \geq 0$$

$$= (1-w)\exp\left(\sum_{n=1}^{k} \frac{w^n}{n}\right)$$

$$\therefore |E_k(w)| \leq (1+|w|)\exp\left(\sum_{n=1}^{k} \frac{|w|^n}{n}\right) \rightarrow ①$$

Since,

$\exp(w) \geq 1+|w|$ ⓐ $\left(\because e^w = 1 + w + \frac{w^2}{2} + \cdots \geq 1+w\right)$

and for large $|w|$, we have

$$\frac{|w|^k}{k} \geq \frac{|w|^{k-1}}{k-1} \geq \cdots \geq |w|$$

$$\Rightarrow k\left(\frac{|w|^k}{k}\right) \geq \sum_{k=1}^{n} \frac{|w|^n}{n}$$

$$\Rightarrow |w|^k \geq \sum_{k=1}^{n} \frac{|w|^n}{n}$$

$\left[\because |w| + \frac{|w|^2}{2!} + \cdots \leq \frac{|w|^k}{k} + \cdots \leq k\frac{|w|^k}{k}\right]$

$$\Rightarrow \exp(|w|^k) \geq \exp\left(\sum_{n=1}^{k} \frac{|w|^n}{n}\right) \rightarrow ⓑ$$

Using ⓐ and ⓑ in ①,

$$|E_k(\omega)| \le (\exp|\omega|)(\exp|\omega|^k)$$

$$\le (\exp(|\omega|+|\omega|^k))$$

$$\le \exp(|\omega|^k+|\omega|^k)$$

$$\le \exp(2|\omega|^k)$$

$$\le \exp(c|\omega|^\lambda) \longrightarrow (2)$$

where, $c \ge 2$ and $\lambda \ge k$

and $|\omega| \ge 1$, $k \ge 0$.

On the other hand, if $|\omega| \le \frac{1}{2}$ and

$k \ge 0$

then, $|E_k(\omega)| \le \exp(2|\omega|^{k+1})$

and so, $|E_k(\omega)| \le \exp(2|\omega|^\lambda)$

for $\lambda \le k+1$ and $|\omega| \le \frac{1}{2}$

$\longrightarrow (3)$

Again, if $\frac{1}{2} \le |\omega| \le 1$, and $k>0$

then, it can be easily shown that for some constant c,

$$|E_k(\omega)| \le \exp(c|\omega|^{k+1})$$

since, $|\omega| \le 1$

so, $|E_k(\omega)| \le \exp(c|\omega|^\lambda) \rightarrow$ (4)

for, $\lambda \le k+1$ and $\frac{1}{2} \le |\omega| \le 1$

from (2), (3) and (4), we conclude that

$|E_k(\omega)| \le \exp(c|\omega|^\lambda)$ for $k \le \lambda \le k+1$

Now, let

$P(z) = \prod_{n=1}^{\infty} E_p\left(\frac{z}{z_n}\right)$ be the

canonical product and we know that p satisfies the inequality

$p \le \sigma \le p+1$

If $\sigma = p+1$, let $\lambda = p+1$

If $\sigma < p+1$, Let λ satisfies,

$\sigma < \lambda < p+1$

since, $\sigma = \inf\{t>0;\ \sum_{i=1}^{\infty} |z_i|^{-t} < \infty\}$

we conclude that,

$$\sum_{i=1}^{\infty} |z_i|^{-\lambda} < \infty$$

Let $\sum_{i=1}^{\infty} |z_i|^{-\lambda} = a$ (say)

Then, we get that,

$$|P(z)| \leq \prod_{n=1}^{\infty} \exp\left(c \left|\frac{z}{z_n}\right|^{\lambda}\right)$$

$$= \exp\left(c|z|^{\lambda} \sum_{n=1}^{\infty} |z_n|^{-\lambda}\right)$$

$$= \exp\left(ac\, |z|^{\lambda}\right) \longrightarrow (6)$$

which holds for every $\lambda > \sigma$ and $\forall z$

Since, (6) is of the form

$$M(r) \leq \exp(r^{\lambda+\epsilon})$$

By defn,

$$\rho = \inf\{\lambda \geq 0;\ M(r) \leq \exp(r^{\lambda})\}$$

It conclude that,

$$\boxed{\rho \leq \sigma}$$

Hence, $\boxed{\sigma = \rho}$