DATE : / 20
PAGE No

$$M(r) \le \exp(r^{\lambda+\varepsilon})$$

By def^n

$$\rho = \inf\{\lambda \ge 0 : M(r) \le \exp(r^{\lambda})\}$$

It conclude that $\rho \le \sigma$

Hence $\sigma = \rho$ H.P.

03/04

Hadamard's Factorisation Th^m :- If $f(z)$ is an entire function of finite order ρ, then $f(z) = z^m e^{g(z)} P(z)$ where m is the order of zeros at $z=0$, $g(z)$ is a polynomial of degree not exceeding ρ and $P(z)$ is the canonical product associated with the sequence of non-zero zeros of $f(z)$.

Proof :- Firstly State and Prove Weierstrass Factorisation Th^m.

An entire function $f(z)$ can be expressed as $f(z) = z^m e^{g(z)} P(z)$ — (1)
where $P(z)$ is itself an entire function.
Here, we shall use the additional hypothesis, that $f(z)$ is of finite order ρ, to show that $g(z)$ is a polynomial of degree not exceeding ρ.
It is clear that the divison of $f(z)$ by cz^m does not affect either the hypothesis or the conclusion of the th^m and so it is sufficient to consider the representation

$$f(z) = e^{g(z)} P(z)$$ so that

$$e^{g(z)} = \frac{f(z)}{P(z)}$$

$$\Rightarrow \quad |e^{g(z)}| = \left|\frac{f(z)}{P(z)}\right|$$

$$\Rightarrow \quad e^{\operatorname{Re} g(z)} = \left|\frac{f(z)}{P(z)}\right|$$

Taking logarithm, we get

$$\operatorname{Re} g(z) = \log|f(z)| - \log|P(z)| \quad -(2)$$

By def^n of order, it follows that

$|f(z)| \leq \exp(r^{\rho+\varepsilon})$ for sufficiently large $|z| = r$ and all $\varepsilon > 0$

$$\log|f(z)| \leq r^{\rho+\varepsilon} \quad -(3)$$

If σ is the convergence exponent of the nonzero zeros of $f(z)$ then we know $\sigma \leq \rho$

Also by Borel's Th^m,

σ is the order of the canonical product P(

Th^m:- Let $P(z)$ be a canonical product of finite order ρ and $q > 0$ and $\varepsilon > 0$ then for all sufficiently large $|z|$,

$$|z - z_i| > |z_i|^{-q}$$

$$\Rightarrow \quad \log|P(z)| > -|z|^{\rho+\varepsilon}$$

Using this th^m, we have

$$\log|P(z)| > -r^{\sigma+\varepsilon} \quad \text{for large } |z| = r$$

Thus $\quad -\log|P(z)| < r^{\sigma+\varepsilon} \leq r^{\rho+\varepsilon} \quad -(4)$

when $\sigma \leq \rho$ and r is large

From (3) and (4), we get

$\log|f(z)| - \log|P(z)| \leq 2r^{\rho+\varepsilon}$
and thus ② gives
$Re\, g(z) \leq 2r^{\rho+\varepsilon}$, Since r is large

Then by th^m → [If the real part of an entire function $g(z)$ satisfies the inequality $Re\, g(z) < r^{\rho+\varepsilon}$ for every $\varepsilon > 0$ and sufficiently large r, then $g(z)$ is a polynomial of degree not exceeding ρ]
we have
$g(z)$ is a polynomial of degree not exceeding ρ.

Eg:- Using Hadamard's factorization th^m, prove that

$$\sin(\pi z) = \pi z \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)$$

Sol^n:- The zeros of $\sin \pi z$ are at $z = 0, \pm1, \pm2, \ldots$
i.e. nonzero zeros of $\sin\pi z$ are $\pm1, \pm2, \ldots$
Since the series $\sum_{n=1}^{\infty} \frac{1}{n}$ ~~conver~~ diverges

and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges so that $p = 1$ is the least integer s.t. $\sum \left|\frac{1}{\pm n}\right|^{p+1}$ $(n \neq 0)$ converges.

Thus the genus of the canonical product is 1 and thus the canonical product associated with nonzero zeros of $\sin\pi z$ is of the form

$$P(z) = \prod_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left(1 - \frac{z}{n}\right) e^{z/n}$$

$$= \prod_{n=1}^{\infty}\left(1 - \frac{z}{n}\right)e^{z/n} \prod_{n=1}^{\infty}\left(1 + \frac{z}{n}\right)e^{-z/n}$$

$$= \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

Now, order of $\sin \pi z$ is 1.
Since $z=0$ is a simple zero of $\sin \pi z$, Hadamard's factorisation of $\sin \pi z$ may be written as

$$\sin \pi z = z e^{g(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

where $g(z)$ is a polynomial of degree not exceeding 1 ie. order of $\sin \pi z$.
Let $g(z) = a_0 + a_1 z$

$$\therefore \sin \pi z = z e^{a_0 + a_1 z} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

<u>To find a_0 and a_1</u>

$$\frac{\sin \pi z}{z} = e^{a_0 + a_1 z} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) \quad - (*)$$

$$\frac{\sin \pi z}{\pi z} = \frac{1}{\pi} e^{a_0 + a_1 z} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) \quad - (1)$$

Since $\frac{\sin \pi z}{\pi z} \to 1$ as $z \to 0$

So making $z \to 0$ in (1), we get

$$1 = \frac{1}{\pi} e^{a_0}$$

$$\Rightarrow e^{a_0} = \pi \quad (**)$$

By (*) and (**)

$$\frac{\sin \pi z}{z} = \pi e^{a_1 z} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) \quad - (2)$$

Replace z by $-z$ in (2) we get

$$\frac{\sin \pi z}{z} = \pi e^{-a_1 z} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) \quad - (3)$$

DATE: / 20
PAGE No

By (2) and (3)

$$e^{a_1 z} = e^{-a_1 z}$$

$$e^{2a_1 z} = 1$$

$$2a_1 z = 0 \Rightarrow a_1 = 0$$

Hence $\sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$