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Section - IV

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f^n **Indefinite Integral:-** If f is an integrable function on $[a,b]$ then f is integrable on any interval $[a,x] \subseteq [a,b]$.
The function F given by
$$F(x) = \int_a^x f(t)\,dt + c \quad \text{, where } c \text{ is a constant}$$
called the indefinite integral of f.

Theorem (I) Let f be an integrable function on $[a,b]$. Then the indefinite integral of f is a cts. function of bdd. variation on $[a,b]$.

Proof:- We know that the indefinite integral is given by
$$F(x) = \int_a^x f(t)\,dt + c$$

Let $x_0 \in [a,b]$
$$F(x_0) = \int_a^{x_0} f(t)\,dt + c$$

Now
$$|F(x) - F(x_0)| = \left| \int_a^x f(t)\,dt + c - \int_a^{x_0} f(t)\,dt - c \right|$$
$$= \left| \int_a^x f(t)\,dt + \int_{x_0}^a f(t)\,dt \right|$$
$$= \left| \int_{x_0}^x f(t)\,dt \right|$$

$$\Rightarrow |F(x) - F(x_0)| \leq \int_{x_0}^x |f(t)|\,dt \qquad -(1)$$

Since f is integrable $\Rightarrow$ $|f|$ is integrable over $[a,b]$
So for given $\varepsilon > 0$, $\exists$ $\delta > 0$ s.t.
for every mble set $A \subseteq E$ with $m(A) < \delta$
$[a,b]$

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then we have
$$\int_A |f| < \epsilon$$

In particular
$$\int_{x_0}^{x} |f(t)|\,dt < \epsilon \quad \text{for } |x-x_0| < \delta$$

i.e. $|F(x) - F(x_0)| < \epsilon$ for $|x-x_0| < \delta$

Hence F is cts. at x_0 [$\because x_0 \in [a,b]$ be arb.]

Hence F is cts. in $[a,b]$

T.S. $F \in BV_a^b$

Let $P = \{a = x_0 < x_1 < \dots < x_n = b\}$ be a Partition of $[a,b]$. Then

$$\sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| = \sum_{i=1}^{n} \left| \int_a^{x_i} f(t)\,dt + c - \int_a^{x_{i-1}} f(t)\,dt - c \right|$$

$$= \sum_{i=1}^{n} \left| \int_a^{x_i} f(t)\,dt + \int_{x_{i-1}}^{a} f(t)\,dt \right|$$

$$= \sum_{i=1}^{n} \left| \int_{x_{i-1}}^{x_i} f(t)\,dt \right| \le \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} |f(t)|\,dt$$

$$= \int_{x_0}^{x_1} |f(t)|\,dt + \int_{x_1}^{x_2} |f(t)|\,dt + \dots + \int_{x_{n-1}}^{x_n} |f(t)|\,dt$$

$$= \int_{x_0}^{x_n} |f(t)|\,dt = \int_a^b |f(t)|\,dt$$

$$\Rightarrow t_a^b(F) \le \int_a^b |f(t)|\,dt < \infty$$ [$\because f$ is integrable $\Rightarrow |f|$ is integrable]

Taking sup.

$\Rightarrow T_a^b(F) < \infty$

$\Rightarrow F \in BV_a^b$ H.P.

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Ex: Theorem Let f be an integrable function on $[a,b]$. If $\int_a^x f(t)\,dt = 0$ — (✱) $\forall\ x \in [a,b]$

then $f = 0$ a.e. in $[a,b]$.

Proof: Let if possible, $f \neq 0$ a.e. in $[a,b]$

Suppose

$E = \{x : f(x) > 0\}$ with $m(E) > 0$

Then $\exists$ a closed set $F \subset E$ with $m(F) > 0$

Let $A = (a,b) - F$

or $A \cup F = (a,b)$

Then A is an open set and we have

$$0 = \int_a^b f(t)\,dt \quad [\text{By } (✱)]$$

$$0 = \int_{A \cup F} f(t)\,dt = \int_A f(t)\,dt + \int_F f(t)\,dt$$

$$\Rightarrow \int_A f(t)\,dt = -\int_F f(t)\,dt \quad (✱✱)$$

But $f > 0$ on F with $m(F) > 0$ $\left[\because f > 0 \text{ on } E \text{ and } F \subset E\right.$

$$\Rightarrow \int_F f(t)\,dt > 0 \quad \Rightarrow \int_F f(t)\,dt \neq 0$$

$$\therefore \int_A f(t)\,dt \neq 0 \quad [\text{By } (✱✱)]$$

But, since A is an open set, therefore it can be expressed as a union of countable collection $\{(a_n, b_n)\}$ of disjoint open intervals.

Thus

$$0 \neq \int_A f(t)\,dt = \int_{\bigcup_{n=1}^{\infty}(a_n,b_n)} f(t)\,dt = \sum_{n=1}^{\infty} \int_{a_n}^{b_n} f(t)\,dt$$

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$\Rightarrow \int_{a_n}^{b_n} f(t)\,dt \neq 0$, for some n

$\Rightarrow$ either $\int_a^{a_n} f(t)\,dt \neq 0$ or $\int_a^{b_n} f(t)\,dt \neq 0$

In either case, we see that if f is positive on a set of positive measure, then for some $x \in [a,b]$ we have

$$\int_a^x f(t)\,dt \neq 0$$

Similar assertion is obtained if f is negative on a set of positive measure.
which is contradiction to given condition
Hence $f = 0$ a.e.

[Then $\because$ If $f = 0$ a.e. holds then
$m\{x : f(x) \neq 0\} = 0$
$E = \{x : f(x) > 0\} \subseteq \{x : f(x) \neq 0\}$
$\Rightarrow m(E) = 0$
But, since we have taken $f \neq 0$ a.e.
Hence we may take. clearly $m(F) > 0$]

Theorem-II. 2016 Let f be a bounded and measurable function defined on $[a,b]$. If

$$F(x) = \int_a^x f(t)\,dt + F(a)$$

then $F'(x) = f(x)$ a.e. in $[a,b]$.

Proof. Since f is bounded and measurable
$\therefore f$ is integrable
Then by theorem I
F is a continuous function F of bounded variation on $[a,b]$
and hence F' exists a.e. in $[a,b]$
[By corollary $\rightarrow$ If f is a function of

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bdd. variation on $[a,b]$, then f' exists a.e. on $[a,b]$]

Since f is bdd.
So $\exists$ $K>0$ s.t.
$$|f(x)| \le K$$

Define
$$f_n(x) = \frac{F(x+h)-F(x)}{h} \quad \text{with } h=\frac{1}{n}$$

Then
$$f_n(x) = \frac{1}{h}\left[\int_a^{x+h} f(t)\,dt + F(a) - \int_a^{x} f(t)\,dt - F(a)\right]$$
$$= \frac{1}{h}\left[\int_a^{x+h} f(t)\,dt + \int_x^{a} f(t)\,dt\right] = \frac{1}{h}\int_x^{x+h} f(t)\,dt$$

$$|f_n(x)| = \frac{1}{h}\left|\int_x^{x+h} f(t)\,dt\right| \le \frac{1}{h}\int_x^{x+h} |f(t)|\,dt \le \frac{K}{h}\int_x^{x+h} dt$$

$$\Rightarrow \quad |f_n(x)| \le K$$

$\Rightarrow$ $\{f_n\}$ is a sequence of bdd. mble function

Now
$$\lim_{n\to\infty} f_n(x) = \lim_{h\to 0} \frac{F(x+h)-F(x)}{h} \qquad \left[\begin{array}{l} h=\frac{1}{n} \\ n\to\infty \Rightarrow h\to 0 \end{array}\right.$$
$$= F'(x)$$

$\Rightarrow$ $\lim_{n\to\infty} f_n(x) = F'(x)$ and $F'(x)$ exists a.e.

Thus $\{f_n\}$ is a seq. of bdd. mble function defined on $[a,b]$ and $|f_n(x)| \le K$ $\forall$ n, x and $\lim_{n\to\infty} f_n(x) = F'(x)$ a.e. for each $x \in [a,b]$

If $c \in [a,b]$ is arbitrary, then

By Bounded Convergence Theorem

[Let $\{f_n\}$ be a sequence of measurable functions defined on a set E of finite measure. Suppose there is a real number M s.t. $|f_n(x)| \leq M$ $\forall x$ and $\forall n$.
If $f(x) = \lim_{n\to\infty} f_n(x)$ for each $x \in E$.

Then $\int_E f = \lim_{n\to\infty} \int_E f_n$]

we have

$$\int_a^c F'(x)\,dx = \lim_{n\to\infty} \int_a^c f_n(x)\,dx = \lim_{h\to 0} \frac{1}{h} \int_a^c [F(x+h) - F(x)]\,dx$$

$$= \lim_{h\to 0} \frac{1}{h} \left[\int_a^c F(x+h)\,dx - \int_a^c F(x)\,dx \right]$$

$$= \lim_{h\to 0} \frac{1}{h} \left[\int_{a+h}^{c+h} F(x)\,dx - \int_a^c F(x)\,dx \right]$$

$$= \lim_{h\to 0} \frac{1}{h} \left[\int_{a+h}^{c} F(x)\,dx + \int_c^{c+h} F(x)\,dx - \int_a^{a+h} F(x)\,dx - \int_{a+h}^{c} F(x)\,dx \right]$$

$$= \lim_{h\to 0} \frac{1}{h} \left[\int_c^{c+h} F(x)\,dx - \int_a^{a+h} F(x)\,dx \right] \quad -(1)$$

Since F is cts. Function
and every continuous function is Riemann integrable
$\therefore$ F is Riemann Integrable.

Then

$$\lim_{h\to 0} \frac{1}{h} \int_c^{c+h} F(x)\,dx = \lim_{h\to 0} \frac{1}{h}\, R\int_c^{c+h} F(x)\,dx$$

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$$\Rightarrow \lim_{h\to 0} \frac{1}{h}\int_c^{c+h} F(x)\,dx = \lim_{h\to 0} \frac{1}{h}\cdot h\,F(c+\theta h) \quad ; \; 0\le\theta<1$$

[Using result given below]

$$= F(c)$$

Similarly

$$\lim_{h\to 0} \frac{1}{h}\int_a^{a+h} F(x)\,dx = F(a)$$

Eqn. (I) becomes

$$\int_a^c F'(x)\,dx = F(c)-F(a) = \int_a^c f(x)\,dx \qquad \text{[By statement]}$$

$$\Rightarrow \int_a^c F'(x)\,dx - \int_a^c f(x)\,dx = 0$$

$$\Rightarrow \int_a^c [F'(x) - f(x)]\,dx = 0 \quad \forall\, c\in[a,b]$$

$$\Rightarrow F'(x) - f(x) = 0 \quad \text{a.e.} \qquad \text{[By previous Result]}$$

Hence $F'(x) = f(x)$ a.e.

Result:- If f is cts. on $[a,b]$. Then $\exists\, c\in(a,b)$ s.t. $\int_a^b f\,dx = (b-a)\,f(c)$

ie. $\exists\; 0<\theta<1$ s.t. $c<c+\theta h<c+h$

$$\int_c^{c+h} f(x)\,dx = h.\,f(\theta h+c)$$

~~Same as above~~ 2017

Theorem III Let f be an integrable function on $[a,b]$, and suppose $F(x) = \int_a^x f(t)\,dt + F(a)$

Then $F'(x) = f(x)$ a.e. in $[a,b]$.

Proof:- W.L.O.G. we may assume that $f\ge 0$

Define a sequence $\{f_n\}$ of functions

$f_n : [a,b]\to\mathbb{R}$, where

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$$f_n(x) = \begin{cases} f(x) & \text{if } f(x) \le n \\ n & \text{if } f(x) > n \end{cases}$$

Clearly, each f_n is a bounded and measurable function.

[$\because$ For each x

either $f_n(x) = f(x)$ or $f_n(x) = n$

$|f_n(x)| = |f(x)|$ or $|f_n(x)| = n \le f(x) < |f(x)|$

$\Rightarrow |f_n(x)| \le |f(x)|$]

So by Theorem II, we have

$$\frac{d}{dx}\int_a^x f_n(t)\,dt = f_n(x) \quad \text{a.e.}$$

Also, $f - f_n \geq 0 \ \forall n$ and,

therefore, the function G_n defined by

$$G_n(x) = \int_a^x (f - f_n)(t)\,dt .$$

is an increasing function of x and $G_n \geq 0$

$$\left[\begin{array}{l} \because \ x_1 \le x_2 \\ \Rightarrow [a, x_1] \subseteq [a, x_2] \Rightarrow \int_a^{x_1} (f - f_n) \le \int_a^{x_2} (f - f_n) \\ \Rightarrow G_n(x_1) \le G_n(x_2) \\ \text{and } f - f_n \geq 0 \ \forall x \Rightarrow G_n \geq 0 \end{array} \right]$$

Then by Lebesgue Theorem [Let f be an increasing real-valued function defined on $[a,b]$. Then f is differentiable a.e. and the derivative f' is measurable] we have

G_n' exists a.e. and $G_n' \geq 0$

Now

$$F(x) = \int_a^x f(t)\,dt + F(a) = \int_a^x (f(t) - f_n(t))\,dt + \int_a^x f_n(t)\,dt + F(a)$$

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$$F(x) = G_n(x) + \int_a^x f_n(t)\,dt + F(a)$$

it follows that $F'(x) = G_n'(x) + f_n(x)$ a.e.

$\Rightarrow F'(x) \geq f_n(x)$ a.e., $\forall\ n$

Since n is arbitrary

$\therefore$ we have

$F'(x) \geq f(x)$ a.e. — (*) $\quad F'(x) \geq \lim_{n\to\infty} f_n(x) = f(x)$

$$\Rightarrow \int_a^b F'(x)\,dx \geq \int_a^b f(x)\,dx = F(b) - F(a) \quad — (\ast\ast)$$

$$\Rightarrow \int_a^b F'(x)\,dx \geq F(b) - F(a) \quad — (1)$$

By Lebesgue Theorem, we have

$$\int_a^b F'(x) \leq F(b) - F(a) \quad — (2)$$

From (1) and (2)

$$\int_a^b F'(x)\,dx = F(b) - F(a) = \int_a^b f(x)\,dx$$

$$\Rightarrow \int_a^b (F'(x) - f(x))\,dx = 0$$

Since $F'(x) - f(x) \geq 0$ a.e. By (*)

Hence $F'(x) - f(x) = 0$ a.e.

$\Rightarrow F'(x) = f(x)$ a.e.

Def^n Lebesgue Point :— Let f be an integrable function on $[a,b]$. A point $x \in [a,b]$ is S.T.B. a Lebesgue point of 'f' if

$$\lim_{h\to 0} \frac{1}{h} \int_x^{x+h} |f(t) - f(x)|\,dt = 0$$

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Theorem Let x be a Lebesgue point of the function 'f'. Then the indefinite integral:

$$F(x) = \int_a^x f(t)\,dt + F(a)$$

is differentiable at the point x, and $F'(x) = f(x)$

Proof :- Since x is a Lebesgue point of f

So by def^n

$$\lim_{h\to 0} \frac{1}{h}\int_x^{x+h} |f(t) - f(x)|\,dt = 0$$

Now

$$\frac{F(x+h) - F(x)}{h} - f(x) = \frac{1}{h}\left[\int_a^{x+h} f(t)dt + F(a) - \int_a^x f(t)dt - F(a)\right] - f(x)$$

$$= \frac{1}{h}\int_x^{x+h} f(t)\,dt - f(x)$$

$$= \frac{1}{h}\int_x^{x+h} (f(t) - f(x))\,dt$$

$$\Rightarrow \left|\frac{F(x+h) - F(x)}{h} - f(x)\right| = \left|\frac{1}{h}\int_x^{x+h} (f(t) - f(x))\,dt\right|$$

$$\le \frac{1}{h}\int_x^{x+h} |f(t) - f(x)|\,dt$$

Taking limit $h \to 0$

$$\left|\lim_{h\to 0} \frac{F(x+h) - F(x)}{h} - f(x)\right| \le \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} |f(t) - f(x)|\,dt = 0$$

$$\Rightarrow \quad |F'(x) - f(x)| \le 0$$

But $\quad |F'(x) - f(x)| \ge 0$

$$\Rightarrow \quad |F'(x) - f(x)| = 0$$

$$\Rightarrow \quad F'(x) = f(x)$$

Hence F is differentiable at Lebesgue point.

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Theorem Every point of continuity of an integrable function 'f' is a Lebesgue point of 'f'.

Proof :- Let f be continuous at x_0.

Then, for every $\varepsilon > 0$, there is a $\delta > 0$ s.t.

$$|f(t) - f(x_0)| < \varepsilon \quad , \text{ for } |t - x_0| < \delta$$

For $|h| < \delta$, we have

$$\frac{1}{h}\int_{x_0}^{x_0+h} |f(t) - f(x_0)|\,dt < \varepsilon$$

H.P.

Def^n **Lebesgue Set** :- The set of all Lebesgue points in $[a,b]$ of 'f' is called the Lebesgue set of the function 'f'.

Def^n **Absolute Continuity** :- A real-valued function 'f' defined on an interval $[a,b]$ is absolutely continuous on $[a,b]$, if for each $\varepsilon > 0$ there exists a $\delta > 0$ s.t.

$$\sum_{i=1}^{n} |f(x_i') - f(x_i)| < \varepsilon$$

for every finite pairwise disjoint collection $\{(x_i, x_i') ; i = 1, 2, \ldots n\}$ of open intervals in $[a,b]$ with $\sum_{i=1}^{n} |x_i' - x_i| < \delta$

Remark Every absolutely cts. function is cts.

Pf Let $x_0 \in [a,b]$

Let $\varepsilon > 0$ be given

then $$\sum_{i=1}^{n} |f(x_i') - f(x_i)| < \varepsilon$$

for every finite pairwise collection $\{(x_i, x_i')\}_{i=1}^{n}$ with $\sum_{i=1}^{n} |x_i' - x_i| < \delta$ — (1)

Let $x \in [a,b]$ be s.t. $|x - x_0| < \delta$

Now $\{(x, x_0)\}$ is finite collection

then by eq^n ①

$$|f(x) - f(x_0)| < \varepsilon.$$

Remark :- Sum and diff. of two absolutely cts. functions is again absolutely cts.

Lemma :- If f is absolutely continuous on $[a,b]$, then f is of bounded variation on $[a,b]$.

Proof :- Since f is absolutely continuous on $[a,b]$

$\therefore$ for $\varepsilon = 1$

$\exists$ a $\delta > 0$ s.t.

$$\sum_{i=1}^{n} |f(x_i') - f(x_i)| < 1$$

for every finite collection $\{(x_i, x_i')\}_{i=1}^{n}$ of pairwise disjoint intervals in $[a,b]$ with

$$\sum_{i=1}^{n} |x_i' - x_i| < \delta$$

Now choose a natural number

$$N > \frac{b-a}{\delta}$$

Divide $[a,b]$ by means of points

$$a = c_0 < c_1 < c_2 < \cdots < c_N = b \quad \text{s.t.}$$

$$c_j - c_{j-1} < \delta \quad \text{for } j = 1, 2, \ldots, N$$

Therefore, for every finite collection $\{(x_i, x_i')\}$ of pairwise disjoint subintervals in $[c_{j-1}, c_j]$, we have

$$\sum_{i} |f(x_i') - f(x_i)| < 1$$

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$\Rightarrow \quad t_{c_{j-1}}^{c_j} \leq 1 \qquad ; \; j = 1, 2, \ldots N$

Taking supremum, we have

$$T_{c_{j-1}}^{c_j}(f) \leq 1 \qquad ; \; j = 1, 2, \ldots N$$

Hence
$$T_a^b(f) = \sum_{j=1}^{N} T_{c_{j-1}}^{c_j}(f) \leq \sum_{j=1}^{N} 1 = N < \infty$$

$\Rightarrow$ f is a function of bounded variation.

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Theorem:— 2016 A function F is an indefinite Integral iff it is Absolutely cts.

Proof:— First suppose that F is an indefinite integral, say
$$F(x) = \int_a^x f(t)\,dt$$

where 'f' is an integrable function on $[a,b]$

Since 'f' is integrable.
So $|f|$ is integrable.
Let $\varepsilon > 0$ be given

$\because$ If f is non-negative integrable function over a set E. Then given $\varepsilon > 0$, $\exists \; \delta > 0$ s.t. for every set $A \subseteq E$ with $m(A) < \delta$ we have $\int_A f < \varepsilon$

Then for given $\varepsilon > 0$,
$\exists$ a $\delta > 0$ s.t.
for every mble set $A \subset [a,b]$ with $m(A) < \delta$ we have

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$$\int_A |f| < \epsilon \qquad \text{--- (*)}$$

Thus for any finite collection $\{(x_i, x_i')\}_{i=1}^{n}$ of pairwise disjoint open intervals in $[a,b]$ with $\sum_{i=1}^{n} |x_i' - x_i| < \delta$, we have

$$\sum_{i=1}^{n} |F(x_i') - F(x_i)| = \sum_{i=1}^{n} \left| \int_a^{x_i'} f(t)\,dt - \int_a^{x_i} f(t)\,dt \right|$$

$$= \sum_{i=1}^{n} \left| \int_{x_i}^{x_i'} f(t)\,dt \right| \leq \sum_{i=1}^{n} \int_{x_i}^{x_i'} |f(t)|\,dt$$

$$= \int_{x_1}^{x_1'} |f(t)|\,dt + \int_{x_2}^{x_2'} |f(t)|\,dt + \cdots + \int_{x_n}^{x_n'} |f(t)|\,dt$$

$$= \int_{\bigcup_{i=1}^{n}(x_i, x_i')} |f(t)|\,dt \qquad \text{--- (1)}$$

let $A = \bigcup_{i=1}^{n} (x_i, x_i')$

$$m(A) = m\left(\bigcup_{i=1}^{n} (x_i, x_i')\right) = \sum_{i=1}^{n} m(x_i, x_i')$$

$$= \sum_{i=1}^{n} l(x_i, x_i') = \sum_{i=1}^{n} |x_i' - x_i| < \delta$$

$\Rightarrow \quad m(A) < \delta$

By (*)

$$\int_A |f(t)|\,dt < \epsilon$$

i.e. $$\int_{\bigcup_{i=1}^{n}(x_i, x_i')} |f(t)|\,dt < \epsilon$$

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By eqn. (1), we have

$$\sum_{i=1}^{n} |F(x_i') - F(x_i)| < \epsilon$$

for every finite collection $\{(x_i, x_i')\}_{i=1}^{n}$ of pairwise disjoint open intervals in $[a,b]$ with $\sum_{i=1}^{n} |x_i' - x_i| < \delta$

Hence F is absolutely continuous.

Conversely :-

Suppose that F is an absolutely continuous function on $[a,b]$.

Then F is of BV on $[a,b]$ [By Lemma]

Then by Jordan Decomposition Theorem

[A function 'f' defined on $[a,b]$ is of BV iff it can be expressed as a difference of two monotone increasing functions (real valued) on $[a,b]$.]

we may write

$$F = F_1 - F_2$$

where F_1 and F_2 are monotone increasing real valued functions.

Then by Lebesgue theorem

F_1' and F_2' exists a.e. and both are mble s.t. $\int_a^b F_1'(x) \leq F_1(b) - F_1(a)$

and $\int_a^b F_2'(x)\,dx \leq F_2(b) - F_2(a)$

$$F'(x) = F_1'(x) - F_2'(x)$$

Now

$$|F'(x)| = |F_1'(x) - F_2'(x)| \leq |F_1'(x)| + |F_2'(x)|$$
$$= F_1'(x) + F_2'(x) \quad -(8)$$

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$$\Rightarrow \int_a^b |F'(x)|\,dx \le \int_a^b \{F_1'(x)+F_2'(x)\}\,dx$$

$$= \int_a^b F_1'(x)\,dx + \int_a^b F_2'(x)\,dx$$

$$\le F_1(b)-F_1(a)+F_2(b)-F_2(a) < \infty$$

$$\Rightarrow \int_a^b |F'(x)|\,dx < \infty$$

$\Rightarrow$ $|F'(x)|$ is integrable

$\Rightarrow$ $F'(x)$ is integrable.

Hence $F'(x)$ is integrable over $[a,b]$.

Let

$$G(x) = \int_a^x F'(t)\,dt + F'(a) \quad — \text{(*)}$$

Since F' is integrable on $[a,b]$

$\therefore$ G is absolutely cts. on $[a,b]$ [By Ist part]

And so the function

$H = F - G$ is absolutely cts. on $[a,b]$

Since F' is integrable on $[a,b]$

$\therefore$ $G'(x) = F'(x)$ a.e. in $[a,b]$ [By Th^m III]

Now

$H'(x) = F'(x) - G'(x) = 0$ a.e.

Thus H is a absolutely cts. function s.t. $H' = 0$ a.e.

So $H = A$ (constant)

$\Rightarrow$ $F - G = A$

$$F(x) = G(x) + A = \int_a^x F'(t)\,dt + F'(a) + A \quad — \text{(**)}$$

Take $x = a$

$F(a) = F'(a) + A$

Eqn. (**) becomes

$$F(x) = \int_a^x F'(t)\,dt + F(a).$$

Hence F is an indefinite integral.

Another Statement of Converse Part
If F is an Absolutely cts. function on $[a,b]$ then F is an indefinite integral of its derivative and more precisely

$$F(x) = \int_a^x f(t)\,dt + C$$

where $f = F'$ and c is a constant.

Corollary:- Every Absolutely cts. function is indefinite integral of its derivative.

Proof:- Prove converse part of above thm.

Theorem:- Let f and g be integrable functions over $[a,b]$. Suppose

$$F(x) = \int_a^x f(t)\,dt + F(a) \quad \text{and} \quad G(x) = \int_a^x g(t)\,dt + G(a),$$

$\forall\ x \in [a,b]$. Then

$$\int_a^b F(t)g(t)\,dt + \int_a^b f(t)G(t)\,dt = F(b)G(b) - F(a)G(a)$$

Proof:- Since f and g are integrable on $[a,b]$ and we know that indefinite integral of a integrable function is absolutely cts.

$\therefore$ F and G are both absolutely continuous on $[a,b]$

$\Rightarrow$ FG is absolutely continuous.

Hence $\int_a^b (FG)' = (FG)(b) - (FG)(a)$

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[∵ By the converse of Above Th^m
If F is Absolutely cts. Then F' is integrable and $\int_a^x F'(t)\,dt = F(x) - F(a)$
$\forall\, x \in [a,b]$]

Since f and g are integrable on $[a,b]$
∴ $F' = f$ a.e. and $G' = g$ a.e. in $[a,b]$
⇒
$(FG)' = FG' + F'G = Fg + fG$ a.e. in $[a,b]$

$$\Rightarrow \int_a^b (FG)'\,dt = \int_a^b (Fg + fG)\,dt$$

$$\Rightarrow (FG)(b) - (FG)(a) = \int_a^b F(t)g(t)\,dt + \int_a^b f(t)G(t)\,dt$$

Hence,
$$\int_a^b F(t)g(t)\,dt + \int_a^b f(t)G(t)\,dt = F(b)G(b) - F(a)G(a)$$

Corollary :- If f and g are absolutely cts. functions on $[a,b]$, then
$$\int_a^b f(t)g'(t)\,dt + \int_a^b f'(t)g(t)\,dt = f(b)g(b) - f(a)g(a)$$

Proof :- Since f and g are absolutely cts. Then clearly
$$f(x) = \int_a^x f'(t)\,dt + f(a)$$
and
$$g(x) = \int_a^x g'(t)\,dt + g(a)$$

where f', g' are integrable over $[a,b]$
Then by Above th^m
$$\int_a^b f(t)g'(t)\,dt + \int_a^b f'(t)g(t)\,dt = f(b)g(b) - f(a)g(a)$$

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Notion of Banach Spaces

Let X be a linear space or vector space over a field of Real numbers or complex numbers. A norm on X is a real valued function $\|.\|$ on X ie $\|.\|: X \rightarrow \mathbb{R}$ (set) which has the following properties:

i) $\|x\| \geq 0 \quad \forall \; x \in X$

ii) $\|x\| = 0 \iff x = 0$

iii) $\|\alpha x\| = |\alpha| \|x\| \quad \forall \; x \in X$ and $\alpha \in \mathbb{R}$ (or $\mathbb{C}$)

iv) $\|x+y\| \leq \|x\| + \|y\| \quad \forall \; x, y \in X$

Def^n A linear space X equipped with a norm $\|.\|$ on it is called a normed linear space or simply a normed space and is denoted by $(X, \|.\|)$.

Def^n A sequence $\{x_n\}$ in a normed space X is said to converge to an element $x \in X$ if given an $\epsilon > 0$, there is an N (+ve int.) s.t.

$$\|x_n - x\| < \epsilon \quad \forall \; n > N$$

We write $x_n \rightarrow x$ or $\lim_{n \to \infty} x_n = x$

Def^n A sequence $\{x_n\}$ in a normed space X is a Cauchy sequence if given an $\epsilon > 0$, there is an N (+ve int.) such that

$$\|x_n - x_m\| < \epsilon \quad \forall \; n, m > N$$

Note Each convergent sequence in a normed space is a Cauchy sequence but the converse may not be true.

Def^n. A normed space is said to be complete if every Cauchy sequence in it is convergent, i.e. if for every Cauchy sequence $\{x_n\}$ in X, there is an element x in X such that $x_n \to x$

Def^n A complete normed linear space is called a Banach space.

L^p spaces :- Let p be a +ve real no. then a mble function f defined on $[0,1]$ is S.T.B belong to the space L^p or $L^p[0,1]$ if

$$\int_0^1 |f|^p < \infty$$

Examples :- 1) The spaces $\mathbb{R}$ and $\mathbb{C}$ (the real numbers and the complex numbers) are Banach spaces with the norm $\|.\|$ given by $\|x\| = |x|$, $x \in \mathbb{R}$ (or $\mathbb{C}$).

2) The spaces $\mathbb{R}^n$ and $\mathbb{C}^n$ are Banach spaces with the norm given by

$$\|x\| = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}, \quad x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n \text{ (or } \mathbb{C}^n)$$

3) Let p, $1 \le p \le \infty$, be a real number. Denote by l^p the space of all sequences $x = (x_1, x_2, x_3, \dots)$ of scalars such that $\sum_{n=1}^{\infty} |x_n|^p < \infty$. The space l^p is a Banach space with the norm defined by

$$\|x_p\| = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p}$$

4.) The space l^{∞} of all bounded sequences $x = (x_1, x_2, x_3, \ldots)$ of scalars is a Banach space under the norm given by

$$\|x\|_{\infty} = \sup_{n} |x_n|$$

5.) The space $P[a,b]$ of all polynomials defined on $[a,b]$ is a normed space under the norm given by

$$\|x\| = \max_{t \in [a,b]} |x(t)|.$$

But $P[a,b]$ is not a Banach space.

6.) The space $C[a,b]$ of all continuous functions defined on $[a,b]$ is a Banach space under the norm given by

$$\|x\| = \max_{t \in [a,b]} |x(t)|$$

Note: In a normed space, we may proceed an important step further and use the series as follows.

Let $\{x_k\}$ be a sequence in a normed space X. Then we can associate with $\{x_k\}$, the sequence $\{s_n\}$ of partial sums given by $s_n = n \sum_{k=1}^{n} x_k$. And we can discuss the behaviour of the series $\sum_{k=1}^{\infty} x_k$ in regard to its convergence or non-convergence accordingly as the sequence $\{s_n\}$ of its partial sums is so.

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\# If f is a measurable function on E, then $|f|^p$ is so for each p, $-\infty < p < \infty$, $p \neq 0$. Designate by $L^p(E)$, the class of all p-integrable functions over E i.e.

$$L^p(E) = \left\{ f : \int_E |f|^p < \infty \right\}$$

Examples:- 1) Let $E = [0, 16]$ and $f : E \to \mathbb{R}$ be a function defined by $f(x) = (x)^{-1/4}$. Then $f \in L^1(E)$ but $f \notin L^4(E)$.

2) Let $E = \left[0, \frac{1}{2}\right]$ and the function $f : E \to \mathbb{R}$ be defined by

$$f(x) = \left[x \log^2\left(\frac{1}{x}\right)\right]^{-1}. \text{ Then } f \in L^1(E)$$

3) Let $E = (0, \infty)$ and the function $f : E \to \mathbb{R}$ be defined by $f(x) = (1+x)^{-1/2}$.
Then $f \in L^p(E)$ for each p, $2 < p < \infty$

→ $L^p(E)$ space is vector space or linear space

Pf: Let $f, g \in L^p(E)$ and $\alpha \in \mathbb{R}$ or $\mathbb{C}$
We know that

$$|f+g|^p \leq 2^p \max\{|f|^p, |g|^p\}$$
$$\leq 2^p \left(|f|^p + |g|^p\right)$$

$$\Rightarrow \int_E |f+g|^p \leq 2^p \left[\int_E |f|^p + \int_E |g|^p\right]$$
$$< \infty \qquad [\because f, g \in L^p(E)]$$

$$\Rightarrow \int_E |f+g|^p < \infty \Rightarrow (f+g) \in L^p(E)$$

Again $\displaystyle\int_E |\alpha f|^p = \int_E |\alpha|^p |f|^p = |\alpha|^p \int_E |f|^p < \infty$

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$\Rightarrow \alpha f \in L^p(E)$

Hence L^p is a vector space.

Furthermore if $f \in L^p(E)$ then the inequalities

$$\begin{cases} 0 \le f^+ \le |f| \\ 0 \le f^- \le |f| \end{cases}$$

imply that f^+, f^- and $|f|$ are also in $L^p(E)$

Remark If $p=1$, then L^1 or $L^1[0,1]$ consist of precisely the Lebesgue integrable function on $[0,1]$.

i.e. $L^1 = \{f : f \text{ is Lebesgue integrable on } [0,1]\}$

If $f \in \{f : f \text{ is Lebesgue integrable on } [0,1]\}$

then f is integrable

$\Rightarrow f^+$ and f^- are integrable

$\Rightarrow \int_0^1 f^+ < \infty$ and $\int_0^1 f^- < \infty$

$\Rightarrow \int_0^1 |f| = \int_0^1 (f^+ + f^-) = \int_0^1 f^+ + \int_0^1 f^- < \infty$

$\Rightarrow \int_0^1 |f| < \infty \Rightarrow f \in L^1$

Now if $f \in L^1 \Rightarrow \int_0^1 |f| < \infty$

$\Rightarrow \int_0^1 (f^+ + f^-) < \infty \Rightarrow \int_0^1 f^+ < \infty$ and $\int_0^1 f^- < \infty$

$\Rightarrow \int_0^1 f = \int_0^1 (f^+ - f^-) = \int_0^1 f^+ - \int_0^1 f^- < \infty$

$\Rightarrow$ f is integrable.

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Proof The inequality holds trivially if either $\alpha=0$ or $\beta=0$.

Hence we assume that $\alpha>0$ and $\beta>0$

Consider the function ϕ defined for a non-negative real number t by,

$$\phi(t)=(1-\lambda)+\lambda t-t^{\lambda} \quad -(1)$$

Then $\phi'(t)=\lambda-\lambda t^{\lambda-1}$

$\phi''(t)=-\lambda(\lambda-1)t^{\lambda-2}$

Put $\phi'(t)=0$ we get

$\lambda-\lambda t^{\lambda-1}=0$

$\lambda(1-t^{\lambda-1})=0$

$\Rightarrow \quad 1-t^{\lambda-1}=0 \Rightarrow 1=t^{\lambda-1}$

$\Rightarrow \quad t=1$

Now $\phi''(t)$ at $t=1$

$\phi''(t)=-\lambda(\lambda-1)1^{\lambda-2}=-\lambda(\lambda-1)>0$ $\quad \left[\begin{array}{l} 0<\lambda<1 \\ \lambda-1<0 \end{array}\right.$

i.e. $\phi''(t)>0$ at $t=1$

Hence ϕ attains its minimum value at $t=1$

$\Rightarrow \quad \phi(t)\geq\phi(1)$ $\quad$ [$\because$ $\phi(1)$ is minimum]

$\Rightarrow \quad \phi(t)\geq 0 \quad \forall t$ $\quad$ [$\because$ $\phi(1)=0$]

$\Rightarrow \quad (1-\lambda)+\lambda t-t^{\lambda}\geq 0 \quad \forall t$

In particular taking $t=\frac{\alpha}{\beta}$

$$(1-\lambda)+\lambda\frac{\alpha}{\beta}-\left(\frac{\alpha}{\beta}\right)^{\lambda}\geq 0$$

$$(1-\lambda)+\lambda\frac{\alpha}{\beta}\geq\left(\frac{\alpha}{\beta}\right)^{\lambda}$$

$$\frac{(1-\lambda)\beta+\lambda\alpha}{\beta}\geq\left(\frac{\alpha}{\beta}\right)^{\lambda}$$

$$(1-\lambda)\beta+\lambda\alpha\geq\alpha^{\lambda}\beta^{1-\lambda}$$

$$\Rightarrow \quad \alpha^{\lambda}\beta^{1-\lambda}\leq\lambda\alpha+(1-\lambda)\beta$$

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The equality holds only when $t=1$

i.e. $\frac{\alpha}{\beta}=1 \Rightarrow \alpha=\beta$

06/04 2016

Riesz - Holder Inequality :- $(1\le p\le\infty)$:-

Let p and q be non-negative extended real numbers such that $\frac{1}{p}+\frac{1}{q}=1$. Here p and q are conjugate to each other. If $f\in L^p$ and $g\in L^q$, then $f.g\in L^1$ and

$$\int|fg| \le \|f\|_p \|g\|_q \quad —①$$

Equality holds iff, for some non-zero constants A and B, we have

$$A|f|^p = B|g|^q \quad a.e.$$

Proof :- 1) when $p=1$ then $q=\infty$ and the inequality ① holds trivially Since in this case $g\in L^\infty$

$\therefore \|g\|_\infty = M$ (say) $\left[\because \|g\|_\infty = \text{ess sup}|g| = \inf\{M: |g(x)|\le M \text{ a.e.}\}\right.$

$\Rightarrow |g| \le \|g\|_\infty$ a.e.

$\Rightarrow |g|\le M$ a.e.

Now

$|fg| = |f||g| \le M|f|$ a.e.

$\left\{\text{we know if } |f|\le g \text{ a.e. where } g \text{ is integrable then } f \text{ is integrable}\right\}$

Hence $f.g\in L^1$

Integrating, we have

$$\int|fg| \le M\int|f| \le \|g\|_\infty \|f\|_1 = \|g\|_q \|f\|_p$$

$$\Rightarrow \int|fg| \le \|f\|_p \|g\|_q$$

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Case II If $1 < p < \infty$

Then consequently $1 < q < \infty$

Then the inequality is trivial if either $f = 0$ a.e. or $g = 0$ a.e.

Assume that $f \neq 0$ a.e. and $g \neq 0$ a.e.

$\Rightarrow \|f\|_p > 0$ and $\|g\|_q > 0$

Now by applying lemma with

$$\lambda = \frac{1}{p}, \quad \alpha = \left(\frac{|f(t)|}{\|f\|_p}\right)^p, \quad \beta = \left(\frac{|g(t)|}{\|g\|_q}\right)^q \qquad \text{—(A)}$$

we get

$$\left[\left(\frac{|f(t)|}{\|f\|_p}\right)^p\right]^{1/p}\left[\left(\frac{|g(t)|}{\|g\|_q}\right)^q\right]^{1/q} \le \frac{1}{p}\left(\frac{|f(t)|}{\|f\|_p}\right)^p + \frac{1}{q}\left(\frac{|g(t)|}{\|g\|_q}\right)^q$$

$$\frac{|f(t)||g(t)|}{\|f\|_p\|g\|_q} \le \frac{1}{p}\frac{|f(t)|^p}{\|f\|_p^p} + \frac{1}{q}\frac{|g(t)|^q}{\|g\|_q^q}$$

$$\frac{|fg(t)|}{\|f\|_p\|g\|_q} \le \frac{1}{p}\frac{|f(t)|^p}{\|f\|_p^p} + \frac{1}{q}\frac{|g(t)|^q}{\|g\|_q^q}$$

$$\Rightarrow \frac{\int |fg(t)|}{\|f\|_p\|g\|_q} \le \frac{1}{p\|f\|_p^p}\int |f(t)|^p + \frac{1}{q\|g\|_q^q}\int |g(t)|^q \qquad \text{—(2)}$$

Since $f \in L^p$ and $g \in L^q$

So $\int |f|^p < \infty$ and $\int |g|^q < \infty$

So by eq^n (2), we have

$$\frac{\int |fg|}{\|f\|_p\|g\|_q} < \infty \Rightarrow \int |fg| < \infty \Rightarrow fg \in L^1$$

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Again by eq^n (2), we have

$$\frac{\int |fg(t)|}{\|f\|_p \|g\|_q} \le \frac{1}{p\|f\|_p^p}(\|f\|_p)^p + \frac{1}{q\|g\|_q^q}(\|g\|_q)^q$$

$$= \frac{1}{p} + \frac{1}{q} = 1$$

$$\Rightarrow \int |fg| \le \|f\|_p \|g\|_q \quad \text{— (A)}$$

Equality in (A) holds iff $\alpha = \beta$
ie $\alpha = \beta$ a.e.

ie

$$\frac{|f(t)|^p}{\|f\|_p^p} = \frac{|g(t)|^q}{\|g\|_q^q} \quad \text{a.e.}$$

ie. $|f(t)|^p \|g\|_q^q = |g(t)|^q \|f\|_p^p$ a.e.

ie $A\,|f(t)|^p = B\,|g(t)|^q$ a.e.

where $A = \|g\|_q^q$ and $B = \|f\|_p^p$

Here A and B are non-zero functions.

H.P.

2018

<u>Riesz Minkowski Inequality</u>:— Let $1 \le p \le \infty$. Then for every pair $f, g \in L^p$, the following inequality holds:

$$\|f+g\|_p \le \|f\|_p + \|g\|_p$$

<u>OR</u> If f and g are in L^p with $1 \le p \le \infty$ and so is $f+g$ and $\|f+g\|_p \le \|f\|_p + \|g\|_p$

Then equality can hold if there are non-ve constants α, β s.t. $\beta f = \alpha g$

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Proof :- 1) If $p=1$ Then inequality is trivially true.

$\because \quad \|f+g\|_1 = \int|f+g| \le \int|f| + \int|g| = \|f\|_1 + \|g\|_1$

$\Rightarrow \quad \|f+g\|_1 \le \|f\|_1 + \|g\|_1$

2) If $p=\infty$
we know that
$|f+g| \le |f| + |g|$
and $\quad |f| \le \|f\|_\infty$ a.e.
$\quad |g| \le \|g\|_\infty$ a.e
$\Rightarrow \quad |f+g| \le |f| + |g| \le \|f\|_\infty + \|g\|_\infty$ a.e.
$\Rightarrow \quad |f+g| \le \|f\|_\infty + \|g\|_\infty$ a.e.
$\Rightarrow \quad \|f+g\|_\infty \le \|f\|_\infty + \|g\|_\infty$

3) Now we assume $1<p<\infty$
Since L^p is a linear space and $f,g \in L^p$
So $f+g \in L^p$
Also we have

$$|f+g|^p = |f+g|^{p-1}|f+g|$$
$$\le |f+g|^{p-1}(|f|+|g|)$$
$$= |f+g|^{p-1}|f| + |f+g|^{p-1}|g|$$

$\Rightarrow \quad \int|f+g|^p \le \int|f+g|^{p-1}|f| + \int|f+g|^{p-1}|g| \quad$ — (1)

4) Let $1<q<\infty$ be s.t. $\frac{1}{p}+\frac{1}{q}=1$

i.e. q is conjugate of p
then $(p-1)q = p \quad$ — (*)

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Now consider

$$\int\left(|f+g|^{p-1}\right)^{q} = \int |f+g|^{(p-1)q} = \int |f+g|^{p} < \infty$$

$[\because f+g \in L^{p}]$

$\Rightarrow \quad |f+g|^{p-1} \in L^{q}$

Then by Riesz Holder Inequality

$|f+g|^{p-1} \in L^{q}, \ |f| \in L^{p} \Rightarrow |f+g|^{p-1}|f| \in L^{1}$

Similarly $\quad |f+g|^{p-1}|g| \in L^{1}$

Also by Riesz Holder Inequality, we have

$$\int |f+g|^{p-1}|f| \leq \|(f+g)^{p-1}\|_{q} \ \|f\|_{p} \qquad -(2)$$

and $$\int |f+g|^{p-1}|g| \leq \|(f+g)^{p-1}\|_{q} \ \|g\|_{p} \qquad -(3)$$

But

$$\|(f+g)^{p-1}\|_{q} = \left(\int\left|(f+g)^{p-1}\right|^{q}\right)^{1/q} = \int |f+g|^{p/q}$$

$$= \left(\|f+g\|_{p}\right)^{p/q} \qquad -(4)$$

By (2) and (4), we have

$$\int |f+g|^{p-1}|f| \leq \|f\|_{p}\left(\|f+g\|_{p}\right)^{p/q} \qquad (5)$$

By (3) and (4) we have

$$\int |f+g|^{p-1}|g| \leq \|g\|_{p}\left(\|f+g\|_{p}\right)^{p/q} \qquad -(6)$$

using (5), (6) in (1), we get

$$\int |f+g|^{p} \leq \|f\|_{p}\left(\|f+g\|_{p}\right)^{p/q} + \|g\|_{p}\left(\|f+g\|_{p}\right)^{p/q}$$

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$$\int |f+g|^p \le (\|f\|_p + \|g\|_p)\,(\|f+g\|_p)^{p/q}$$

Case I. If $\|f+g\|_p \ne 0$ and finite
then dividing by $(\|f+g\|_p)^{p/q}$ eq (7) we get

$$\frac{\int |f+g|^p}{(\|f+g\|_p)^{p/q}} \le \|f\|_p + \|g\|_p$$

$$\Rightarrow \frac{(\|f+g\|_p)^p}{(\|f+g\|_p)^{p/q}} \le \|f\|_p + \|g\|_p$$

$$\Rightarrow (\|f+g\|_p)^{p(1-\frac{1}{q})} \le \|f\|_p + \|g\|_p$$

$$\Rightarrow \|f+g\|_p \le \|f\|_p + \|g\|_p \qquad \left[\because p\left(1-\tfrac{1}{q}\right)=1\right.$$

Hence the result

Case 2- If $\|f+g\|_p = 0$
Then result holds trivially H.P.

<u>Riesz - Holder Inequality for $0<p<1$</u> :-
Let $0<p<1$
and q be the conjugate exponent of p.
If $f \in L^p$ and $g \in L^q$ then

$$\int |fg| \ge \left(\int |f|^p\right)^{1/p} \left(\int |g|^q\right)^{1/q}$$

ie. $$\int |fg| \ge \|f\|_p \, \|g\|_q$$

provided $\int |g|^q \ne 0$

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Proof first note that $q<0$ since $p<1$ and $\frac{1}{p}+\frac{1}{q}=1$

Put $p=\frac{1}{P}$ and $-\frac{p}{q}=\frac{1}{Q}$

Then $P>1$, $Q>1$ and $\frac{1}{P}+\frac{1}{Q}=1$

Further put

$$|fg| = F^P \quad \text{and} \quad |g|^q = G^Q \quad \Big\} \quad -\ (*)$$

then

$$|f|^p = |f|^{1/P} = |f|^{-q/Q}$$

$$= \left(|f||g||g|^{-1}\right)^{-q/Q} = \left(|fg||g|^{-1}\right)^{-q/Q}$$

$$= |fg|^{-q/Q}\,|g|^{q/Q} = |fg|^p\,|g|^{-p}$$

$$= F^{P.p}\,G^{-pQ/q} = F.G.$$

$$\Rightarrow \quad |f|^p = F.G \qquad -\ (1)$$

Also we can find that

$F \in L^P$, $G \in L^Q$

then by applying Riesz Holder Inequality

$$\int F.G \leq \|F\|_P\,\|G\|_Q \qquad -\ (2)$$

Now integrating eq^n (1)

$$\int |f|^p = \int F.G \leq \|F\|_P\,\|G\|_Q \qquad [\text{using } (2)]$$

$$= \left(\int |F|^P\right)^{1/P}\left(\int |G|^Q\right)^{1/Q}$$

$$= \left(\int |fg|\right)^p\left(\int |g|^q\right)^{-p/q} \qquad [\text{using } (*)]$$

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$$\Rightarrow \int |f|^p \le \left(\int |fg|\right)^p \left(\int |g|^q\right)^{-p/q}$$

$$\Rightarrow \left(\int |fg|\right)^p \ge \int |f|^p \left(\int |g|^q\right)^{p/q}$$

$$\int |fg| \ge \left(\int |f|^p\right)^{1/p} \left(\int |g|^q\right)^{1/q}$$

$$\Rightarrow \int |fg| \ge \|f\|_p \, \|g\|_q$$ H.P.

Note - The condition $\int |g|^q \ne 0$ is necessary since $q<0$

2014

Riesz - Minkowski Inequality for $0<p<1$:- Let $0<p<1$ and f, g be in L^p s.t. $f \ge 0$ and $g \ge 0$. Then $\|f+g\|_p \ge \|f\|_p + \|g\|_p$

Proof :-

We know that

$$(f+g)^p = (f+g)\,(f+g)^{p-1}$$

$$= f(f+g)^{p-1} + g(f+g)^{p-1}$$

$$\Rightarrow \int |f+g|^p = \int |f|\,|f+g|^{p-1} + \int |g|\,|f+g|^{p-1}$$

$[\because f \ge 0, g \ge 0]$

Now following exactly on the lines of the proof of Riesz Minkowski inequality by using Riesz - Holder inequality for $0<p<1$.

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Remark Every convergent sequence in normed space is a Cauchy sequence.

Pf. Let $\{f_n\}$ be a sequence s.t. $f_n \to f$. Then for given $\varepsilon > 0$, $\exists$ a +ve integer N such that

$$\|f_n - f\| < \frac{\varepsilon}{2} \quad \forall \; n \geq N$$

Now if $m \geq N$ then $\|f_m - f\| < \frac{\varepsilon}{2}$

Consider

$$\|f_n - f_m\| = \|f_n - f + f - f_m\| \leq \|f_n - f\| + \|f - f_m\|$$

$$= \|f_n - f\| + \|f_m - f\|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \forall \; n, m \geq N$$

$$\Rightarrow \|f_n - f_m\| < \varepsilon \quad \forall \; n, m \geq N$$

Hence $\{f_n\}$ is a Cauchy sequence.

07/04

Def^n:– A sequence $\{x_k\}$ in a normed space X is S.T.B. summable to the sum s if the sequence $\{s_n\}$ of the partial sums of the series $\sum_k x_k$ converges to s in X, i.e.

$$\|s_n - s\| \to 0 \quad \text{as } n \to \infty$$

or

$$\left\| \sum_{k=1}^{n} x_k - s \right\| \to 0 \quad \text{as } n \to \infty$$

In this case, we write

$$s = \sum_{k=1}^{\infty} x_k .$$

The sequence $\{x_k\}$ is S.T.B absolutely summable if $\sum_{k=1}^{\infty} \|x_k\| < \infty$

Theorem A normed space X is complete if and only if every absolutely summable sequence is summable.

Proof Assume that X is a complete normed space. Let $\{x_n\}$ be an absolutely summable sequence in X. Then

$$\sum_{n=1}^{\infty} \|x_n\| = M < \infty$$

Thus, for each $\varepsilon > 0$, there is an N s.t.

$$\sum_{n=N}^{\infty} \|x_n\| < \varepsilon$$

Let $s_n = \sum_{k=1}^{n} x_k$ be the partial sums of the series $\sum_{k=1}^{\infty} x_k$. Then, we have

$$\|s_n - s_m\| = \left\| \sum_{k=m+1}^{n} x_k \right\|$$
$$\leq \sum_{k=m+1}^{n} \|x_k\|$$
$$\leq \sum_{k=N}^{\infty} \|x_k\| < \varepsilon \quad \text{for } n \geq m > N.$$

Thus $\{s_n\}$ is a Cauchy sequence in X and must converge to some element (say) s in X, since X is complete. Hence $\{x_n\}$ is summable in X.

Conversely :- Suppose each absolutely summable sequence in X is summable in X.

Let $\{x_n\}$ be a Cauchy sequence in X.

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Then for each k, there is an integer n_k s.t.

$$\|x_n - x_m\| < \frac{1}{2^k}, \quad \forall\ n, m > n_k$$

We may choose n_k s.t. $n_{k+1} > n_k$.
Then $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$.
Set
$$y_1 = x_{n_1}$$
$$y_2 = x_{n_2} - x_{n_1}$$
$$\vdots$$
$$y_k = x_{n_k} - x_{n_{k-1}}$$
$$\vdots$$

We note that
1) $$\sum_{i=1}^{k} y_i = x_{n_k}$$
2) $\|y_k\| < \frac{1}{2^k}$, $k > 1$ and as such

$$\sum \|y_k\| \le \|y_1\| + \sum_{k=2}^{\infty} 2^{1-k} = \|y_1\| + 1 < \infty$$

Thus, the sequence $\{y_k\}$ is absolutely summable and hence summable to some element (say) x in X.
It remains to prove that $\lim_{n \to \infty} x_n = x$

Since $\{x_n\}$ is a Cauchy sequence, given an $\varepsilon > 0$, there is an N s.t.

$$\|x_n - x_m\| < \frac{\varepsilon}{2}, \quad \forall\ n, m > N$$

Further, since $x_{n_k} \Rightarrow x$, there is a K s.t.

$$\|x_{n_k} - x\| < \frac{\varepsilon}{2} \quad \forall\ k \geq K$$

Choose k so large that $k > K$ and $n_k > N$.

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Hence

$$\|x_n - x\| = \|x_n - x_{n_k} + x_{n_k} - x\|$$
$$\leq \|x_n - x_{n_k}\| + \|x_{n_k} - x\|$$
$$< \varepsilon \quad \forall \; n > N$$

$\Rightarrow \quad \|x_n - x\| < \varepsilon \quad \forall \; n > N$

$\Rightarrow \quad x_n \to x$ H.P.

$\therefore$ $\{x_n\}$ is cgt. in X

i.e. every Cauchy sequence in X is cgt. in X

$\therefore$ X is complete.

11/04 2016, 2017

Riesz - Fischer Theorem :- The normed spaces L^p, $1 \leq p \leq \infty$ are complete.

Proof :- Assume $1 \leq p < \infty$.

It is enough to show that each absolutely summable sequence in L^p is summable in L^p to some element in L^p.

Let $\{f_n\}$ be a sequence in L^p with

$$\sum_{n=1}^{\infty} \|f_n\|_p = M < \infty$$

Define a sequence $\{g_n\}$ of functions, where

$$g_n(x) = \sum_{k=1}^{n} |f_k(x)|$$

Observe, for each x that $\{g_n(x)\}$ is an increasing sequence of (extended) real numbers and as such must converge to an extended real number $g(x)$ (say),

i.e. $g_n(x) \to g(x)$, for each $x \in [a, b]$

Since the functions g_n are measurable, the function g is so.

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Also, By Riesz Minkowski Inequality for $1 \le p \le \infty$

$$\|g_n\|_p = \left\| \sum_{k=1}^{n} |f_k| \right\|_p \le \sum_{k=1}^{n} \|f_k\|_p < M$$

$$\Rightarrow \left(\int |g_n|^p \right)^{1/p} < M$$

$$\Rightarrow \int |g_n|^p < M^p$$

$\therefore$ since $g_n \geq 0$, By Fatou's Lemma, we have

$$\int g^p \le M^p$$

This verifies that g^p is integrable, and hence $g(x)$ is finite a.e. on $[a,b]$.

Thus, we find that, for each x for which $g(x)$ is finite, the sequence $\{f_n(x)\}$ is an absolutely summable sequence of real numbers, and therefore must be summable to a real number (say) $s(x)$.

[Since for a sequence of real numbers absolute summability implies summability]

Let us set

$s(x) = 0$ for those x where $g(x) = \infty$

Then the function s so defined is the limit a.e. of the partial sums

$$s_n(x) = \sum_{k=1}^{n} f_k(x)$$

i.e. $s_n(x) \rightarrow s(x)$ a.e.

Hence s is a measurable function.

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Further

$$|s_n(x)| = \left| \sum_{k=1}^{n} f_k(x) \right| \le \sum_{k=1}^{n} |f_k(x)|$$

$$= g_n(x) \le g(x) \quad \text{for each } n$$

$\Rightarrow \quad |s_n(x)| \le g(x)$

$\Rightarrow \quad \lim_{n\to\infty} |s_n(x)| \le g(x)$

$\Rightarrow \quad |s(x)| \le g(x) \quad$ a.e.

Therefore $s \in L^p$, since $g \in L^p$,
and

$$|s_n(x) - s(x)|^p \le 2^p \max\{|s_n(x)|^p, |s(x)|^p\}$$
$$\le 2^p (g(x))^p$$

where $2^p g^p$ is an integrable function
and

$|s_n(x) - s(x)|^p \to 0 \quad$ a.e.

So by Lebesgue Dominated Convergence Th^m, we get

$\int |s_n - s|^p \to 0$

$\Rightarrow \quad \|s_n - s\|_p \to 0$

$\Rightarrow \quad s_n \to s$

Hence the sequence $\{f_n\}$ is summable in L^p and has the sum s in L^p.

$\Rightarrow$ L^p is complete for $1 \le p < \infty$

ii) Now for the case $p = \infty$,
Let $\{f_n\}$ be a Cauchy sequence in L^∞.
Then

$|f_n(x) - f_m(x)| \le \|f_n - f_m\|_\infty$

except on a set $A_{n,m} \subset [a,b]$ with
$m(A_{n,m}) = 0$

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If $A = \bigcup_{n,m} A_{n,m}$ then $m(A) = 0$

and $|f_n(x) - f_m(x)| \le \|f_n - f_m\|_\infty$ $\forall$ n and m and $\forall$ $x \in [a,b] - A$

Therefore, it follows that $\{f_n\}$ converges uniformly to a bounded limit f outside A

Since every uniformly cgt. seq. is cgt.

So $\{f_n\}$ converges to f (say) outside A

Clearly $f \in L^\infty$

$$\because \quad |f(x)| = |f(x) - f_n(x) + f_n(x)|$$
$$\le |f(x) - f_n(x)| + f_n(x)$$
$$< \epsilon + |f_n(x)|$$
$$\le \epsilon + \|f_n\|_\infty \quad \text{a.e}$$

Hence f is bounded a.e.

ie $f \in L^\infty$

So every Cauchy seq. is convergent

ie L^∞ is complete space

Hence L^p is complete for $1 \le p \le \infty$ H.P.

<u>Bounded Linear Functionals on L^p spaces</u>:-

We define a linear functional on a normed linear space X to be a mapping F of the space X into the set of Real no. such that

$$F(\alpha f + \beta g) = \alpha F(f) + \beta F(g)$$

where α, β are scalars.

We say that linear functional is bounded if $\exists$ a constant M such that

$$|F(f)| \le M\|f\| \quad - (1) \quad \forall f \in X$$

the smallest M for which (1) is true

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$$\|F\| = \sup_{0 \neq f \in L^p} \frac{|F(f)|}{\|f\|_p} \leq \sup_{0 \neq f \in L^p} \frac{\|f\|_p \|g\|_q}{\|f\|_p} \quad [\text{Using } ④]$$

$$= \|g\|_q$$

$$\Rightarrow \|F\| \leq \|g\|_q \quad -③$$

Hence F is a Bounded linear functional on L^p s.t. $\|F\| \leq \|g\|_q$

Now consider

$$f = |g|^{q/p} . \operatorname{sgn} g$$

$$|f|^p = \left| |g|^{q/p} \operatorname{sgn} g \right|^p = \left| |g|^{q/p} \right|^p |\operatorname{sgn} g|^p$$

$$= |g|^q . 1 = |g|^q$$

$$\Rightarrow |f|^p = |g|^q$$

Also

$$fg = |g|^{q/p} \operatorname{sgn} g . g = |g|^{q/p} |g| .$$

$$= |g|^{q/p+1} = |g|^q$$

Thus $fg = |g|^q = |f|^p$ ④④

Now

$$\left(\int |f|^p \right)^{1/p} = \left(\int |g|^q \right)^{1/p}$$

$$= \left(\|g\|_q \right)^{q/p} < \infty \quad [\because g \in L^q$$

$$\Rightarrow f \in L^p$$

and $\|f\|_p = \left(\|g\|_q \right)^{q/p}$

Now

$$F(f) = \int fg = \int |g|^q = \left(\|g\|_q \right)^q$$

$$= \|g\|_q . \|f\|_p$$

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$$\left\{ \because \|g\|_q \cdot \|f\|_p = \|g\|_q \cdot (\|g\|_q)^{q/p} = (\|g\|_q)^{1+\frac{q}{p}} = (\|g\|_q)^q \right\}$$

Hence

$$\|F\| = \sup_{0 \neq f \in L^p} \frac{|F(f)|}{\|f\|_p} \geq \frac{|F(f)|}{\|f\|_p}$$

$$= \frac{(\|g\|_q)^q}{(\|g\|_q)^{q/p}} = (\|g\|_q)^{q(1-\frac{1}{p})}$$

$$\|F\| \geq (\|g\|_q)^{q \cdot \frac{1}{q}}$$

Thus $\|F\| \geq \|g\|_q$ — (4)

By (3) and (4), we get

$$\|F\| = \|g\|_q$$

Hence Proved.

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2017

Theorem : Jensen's Inequality : If ϕ is convex function on $(-\infty, \infty)$ and f is integrable function on $[0,1]$ then

$$\int \phi\,(f(t))\,dt \geq \phi\left[\int f(t)\,dt\right]$$

Proof : Let $x_0 = \int_0^1 f(t)\,dt$

and let $y = m(x - x_0) + \phi(x_0)$ be equation of supporting line at x_0

Then $m\,(f(t) - x_0) + \phi(x_0) \leq \phi\,(f(t))$

so that integrating both sides with respect to t over $[0,1]$, we get

$$\phi\,(x_0) \leq \int_0^1 \phi\,(f(t))\,dt$$

$$\Rightarrow \phi\left\{\int_0^1 f(t)\,dt\right\} \leq \int_0^1 \phi\,(f(t))\,dt$$

This complete the proof of theorem

Cor (i) If f is an integrable function on $[0,1]$, then

$$e^{\int_0^1 f(t)\,dt} \leq \int_0^1 e^{f(t)}\,dt$$

Proof : It follows from Jensen's inequality by taking

$$\phi(x) = e^x \quad (-\infty < x < \infty)$$

Remark (i) Geometrical interpretation of Jensen's inequality :

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Since the pt. $\lambda x_1 + (1-\lambda)x_2$ is centroid of masses λ and $1-\lambda$ at x_1 and x_2. Therefore a function is convex if its values at the centroid of a two pts mass is less than the weighted average of its values at two pts.

Jensen's inequality is a generalisation of this fact i.e. if we define a mass distribution μ on the line by taking

$$\mu(a,b) = m\{t : a < f(t) \leq b\}$$

then $\int f(t)\,dt$ is the centroid of this mass and $\int \phi(f(t))\,dt = \int \phi(x)\,d\mu$ is weighted average of ϕ.

ϕ is strictly convex if

$$\phi[\lambda x + (1-\lambda)y] < \lambda\phi(x) + (1-\lambda)\phi(y)$$

$\forall$ $x, y \in (a,b)$ and all $\lambda \in (0,1)$.

2014

Theorem 2: If F is an absolutely cts function on $[a,b]$, then F is an indefinite integral of its derivative

Or

If F is an absolutely cts function on $[a,b]$ then F' is an integrable over $[a,b]$ and $\int_a^x F'(t)\,dt = F(x) - F(a)$

Proof: Let F be an absolutely cts function on $[a,b]$

$\therefore$ F is of BV on $[a,b]$

$\therefore$ By Jordan Decomposition Theorem, $F = F_1 - F_2$, where F_1 and F_2 are monotonically ↑ing function

$\Rightarrow$ F_1' and F_2' exist a.e. on $[a,b]$

$\Rightarrow$ $F' = F_1' - F_2'$ exist a.e. on $[a,b]$

Now
$$F' \le |F'| = |F_1' - F_2'| \le |F_1'| + |F_2'|$$

$$\therefore \int_a^b |F'| \le \int_a^b |F_1'| + \int_a^b |F_2'|$$

$$\le F_1(b) - F_1(a) + F_2(b) - F_2(a)$$

$$< \infty \quad \text{[By Lebesgue Theorem]}$$

$\Rightarrow$ F' is integrable over $[a,b]$

Let $G(x) = \int_a^x F'(t)\,dt$

Then $G(x)$ is absolutely continuous function on $[a,b]$

Let $H = F - G$

Then H is absolutely continuous and

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$$H' = F' - G' = 0 \quad a.e$$

$\Rightarrow \quad H = c$ where c is a constant

$\therefore \quad F = G + H = G + c$

i.e. $$F(x) = \int_a^x F'(t)\,dt + c$$

Putting $x = a$, we get $F(a) = c$

$$\therefore \quad F(x) = \int_a^x F'(t)\,dt + F(a)$$

Convex functions :- A function ϕ defined on an open interval (a, b) is said to be convex if for any λ, $0 \le \lambda \le 1$ and $x, y \in (a, b)$ we have

$$\phi\{\lambda x + (1-\lambda)y\} \le \lambda\phi(x) + (1-\lambda)\phi(y)$$

The end points a, b can take the values $+\infty$, $-\infty$ respectively. Geometrically the definition says the line segement joining the points $X = (x, \phi(x))$ and $Y = (y, \phi(y))$ is always above the graph of ϕ in R^2

orem :- If ϕ is convex on (a, b), x, y, x', $y' \in (a, b)$ such that $x \le x' < y'$ and $y \le y \le y'$ then

$$\frac{\phi(y) - \phi(x)}{y - x} \le \frac{\phi(y') - \phi(x')}{y' - x'}$$

i.e. the line segment joining the pt. $X' = (x', \phi(x'))$ and $Y' = (y', \phi(y'))$ has a larger slope than the line

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segment joining the pts $X = (x', \phi(x'))$ and $Y = (y, \phi(y))$

roof :- Let $x < y \le y'$ and $\lambda = \frac{y-x}{y'-x}$

Then $0 < \lambda \le 1$ and $\lambda(y'-x) = y-x$

$$\lambda y' - \lambda x = y - x$$
$$\lambda y' - \lambda x + x = y$$
$$y = \lambda y' + (1-\lambda)x$$

so that convexity of ϕ yields that

$$\phi(y) = \phi(\lambda y' + (1-\lambda)x)$$
$$\le \lambda\, \phi(y') + (1-\lambda)\phi(x)$$
$$\Rightarrow \frac{\phi(y) - \phi(x)}{y-x} \le \frac{\phi(y') - \phi(x)}{y'-x} \quad -(1)$$

Similarly $x \le x' < y'$

$$\Rightarrow \frac{\phi(y') - \phi(x)}{y'-x} \le \frac{\phi(y') - \phi(x')}{y'-x'} \quad -(2)$$

From (1) and (2) we get

$$\frac{\phi(y) - \phi(x)}{y-x} \le \frac{\phi(y') - \phi(x')}{y'-x'}$$

Hence the proof.

2019, 2018

eorem If ϕ is differeniable on (a,b) and ϕ' is non ding on (a,b) then ϕ is convex function OR

If ϕ is cts function on (a,b) and if one derivative (say D^+) of ϕ is non ding then ϕ is convex.

of :- Given x, y with $a < x < y < b$

Define a function ψ on $[0,1]$ by

$$\psi(t) = \phi[ty + (1-t)x] - t\phi(y) - (1-t)\phi(x)$$
$$\le 0$$

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Claim :- ψ is non positive on $[0,1] \Rightarrow \psi \le 0$ on $[0,1]$

Now ψ is cts and $\psi(0) = \psi(1) = 0$

Moreover

$$D^+\psi = (y-x)D^+ - \phi(y) + \phi(x)$$

$\Rightarrow$ $D^+\psi$ is non ↓ing on $[0,1)$

Let c be the pt where ψ assume its maximum on $[0,1)$

If $c = 1$ then $\psi(t) \le \psi(1) = 0$ on $[0,1)$

Suppose $c \in [0,1)$

Since ψ has a local maximum at c

$\therefore$ $D^+(\psi)(a) \le 0$

But $D^+\psi \le 0$ on $[0,c]$

Consequently ψ is non ↑ing on $[0,c]$

$\Rightarrow$ $\psi(c) \le \psi(0) = 0$

Therefore the maximum of ψ on $[0,1]$ is non positive and $\psi \le 0$ on $[0,$

$\Rightarrow$ $\phi[ty + (1-t)x] \le t\cdot\phi(y) + (1-t)\phi(x)$

$\Rightarrow$ ϕ is convex function

Cor (i) If ϕ has 2nd order derivative at each pt of (a,b) then ϕ is convex on (a,b) iff $\phi''(x) \ge 0$ for each $x \in (a,b)$

Proof :- If ϕ is convex f^n on (a,b) and $x_0 \in (a,$ then the line $y = m(x-x_0) + \phi(x_0)$ passing through the $(x, \phi(x_0))$ is called supporting line at x_0 if

$$\phi(x) \ge m(x-x_0) + \phi(x_0)$$

i.e. if the graph of ϕ always lies above the line

Result :- The line $y = m(x-x_0) + \phi(x_0)$ is supporting line at x_0 iff its slope m lies between the left and right hand derivative of ϕ at x_0

Thus in particular there is always atleast on supporting line at each pt.

Theorem Differentiation of Monotone Functions

Definition :- Let $E \subseteq R$ Then a collection $\mathcal{J}$ of intervals is said to be vitali cover of E. For given $\varepsilon > 0$ and any $x \in E$, $\exists$ $I \in \mathcal{J}$ such that $x \in I$ and $l(I) < \varepsilon$

e.g. If $\{r_n\}$ is a set of rationals in $[a, b]$, then collection $\{I_{n,i}\}$ where $I_{n,i} = \left[r_n - \frac{1}{i}, r_n + \frac{1}{i} \right]$, $n, i \in N$ forms a vitali cover of $[a, b]$

2018 bab

Theorem Vitali's covering Lemma :- Let E be a set of finite measure and $\mathcal{J}$ be a vitali cover of E. Then for given $\varepsilon > 0$, $\exists$ a finite disjoint collection $\{I_1, I_2, \ldots I_N\}$ of intervals of $\mathcal{J}$ such that

$$m^* \left(E - \bigcup_{n=1}^{N} I_n \right) < \varepsilon$$

Proof :- It is sufficient to prove the result in case each interval in $\mathcal{J}$ is closed because otherwise we replace each interval by its closure and observe that set of end points of $I_1, I_2, \ldots I_n$ has measure zero. Let O be an open set such that $O \supseteq E$ and

$$m^* O < m^* E + 1 < \infty$$

Since $\mathcal{J}$ is vitali cover of E, we may assume that each interval

in $\mathcal{J}$ is contained in O, otherwise we would remove from $\mathcal{J}$ all those intervals which extend beyond O and the remaining intervals in $\mathcal{J}$ still form a vitali cover of E.
We now use induction to find a sequence $\{I_n\}$ of disjoint interval of of $\mathcal{J}$ as follows:
Let I_1 be an arbitary interval of $\mathcal{J}$ and η_1 be the supremum of the length of the intervals in $\mathcal{J}$ which do not have any pt. in common with I_1
Then $\eta_1 < \infty$ as $\eta_1 \leq mO < \infty$
We then choose an interval I_2 from $\mathcal{J}$ disjoint from I_1 such that

$$l(I_2) > \frac{1}{2}\eta_1$$

Let η_2 be supremum of length of intervals in $\mathcal{J}$ which do not have any pt. in common with I_1 or I_2 and $\eta_2 < \infty$

choose I_3 from $\mathcal{J}$ which is disjoint from $I_1 \cup I_2$ such that

$$l(I_3) > \frac{1}{2}\eta_2$$

Having chosen n disjoint intervals $I_1, I_2, \ldots I_n$ we denote $\eta_n < \infty$, the supremum of length of all intervals in $\mathcal{J}$ which do not have point common with $\bigcup_{i=1}^{n} I_i$ and choose an

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interval I_{n+1} from $\mathcal{J}$ such that it is disjoint from preceding intervals and
$$l(I_{n+1}) > \frac{1}{2}\eta_n$$
}

Thus we have a sequence $\{I_n\}$ of disjoint intervals $\mathcal{J}$ such that
$$I_i \cap I_j = \phi \quad \text{for } i \neq j$$
and $l(I_{n+1}) > \frac{1}{2}\eta_n$, $\eta_n < \infty \ \forall\ n$

and $\{\eta_n\}$ is monotonically ↓ing sequence of real numbers

$$\therefore \bigcup_{n=1}^{\infty} I_n \subseteq O$$

$$\therefore \sum_{n=1}^{\infty} l(I_n) \leq mO < \infty$$

$\Rightarrow$ For given $\varepsilon > 0$, $\exists$ $N \in \mathbb{N}$ such that
$$\sum_{n=N+1}^{\infty} l(I_n) < \frac{\varepsilon}{5} \quad -①$$

Let $$J = E - \bigcup_{n=1}^{N} I_n$$

Claim $$m^*(J) < \varepsilon$$

Let $x \in J$ be any arbitary element
$$\Rightarrow x \notin \bigcup_{n=1}^{N} I_n$$

Since $\bigcup_{n=1}^{N} I_n$ is closed set not containing x, we can find an interval I in $\mathcal{J}$ such that $x \in I$ and $l(I)$ is so small that
$$I \cap I_i = \phi, \quad i = 1, 2, \ldots, N$$
$$\therefore l(I) = \eta_N < 2\, l(I_{N+1})$$

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Since $\lim_{n \to \infty} l(I_n) = 0$, therefore I must be atleast one of intervals of sequence $\{I_n\}$

Let n_0 be smallest integer such that I meets I_{n_0}
Then $n_0 > N$ and $l(I) \leq \eta_{n_0-1} < 2 l($

Since $x \in I$ and I has a pt in common with I_{n_0}, therefore distance of x from mid point of I_{n_0} is at most

$$l(I) + \frac{1}{2} l(I_{n_0}) < 2 l(I_{n_0}) + \frac{1}{2} l($$

$$= \frac{5}{2} l(I_{n_0}) \quad [\text{By } ②$$

$\therefore$ If J_{n_0} is an interval concentric with I_{n_0} such that $l(J_{n_0}) = 5 l(I$
then $x \in J_{n_0}$
i.e. $\forall \ x \in J$, $\exists \ n \geq N+1$ such that
$x \in J_n$ and $l(J_n) = 5 l(J_n)$

$$\Rightarrow \quad J \subseteq \bigcup_{n=N+1}^{\infty} J_n$$

$$\Rightarrow \quad m^* J \leq \sum_{n=N+1}^{\infty} l(J_n) = 5 \sum_{n=N+1}^{\infty} l(I_n) < \infty$$

Hence the proof. [By ①]