12-2-20

2013, 14, 16, 17

Section - 4

Topic: Integral Perturbation Method:

Let $f(P) = \int_S K(P,Q)\, g(Q)\, ds$ ——— ① $P \in S$

be an fredhlom integral eqn of 1st kind with $\vec{P} = \vec{x}$ & $\vec{Q} = \vec{\xi}$, on the surface S

we will discuss a method based on perturbation technique which gives an approximate soln

Let ϵ be a perturbation parameter occuring in eqn ① then we expend all the functions K, f and g in power of ϵ as

$$K = K_0 + \epsilon K_1 + \epsilon^2 K_2 + \cdots \quad \text{——— ②}$$

$$f = f_0 + \epsilon f_1 + \epsilon^2 f_2 + \cdots \quad \text{——— ③}$$

$$g = g_0 + \epsilon g_1 + \epsilon^2 g_2 + \cdots \quad \text{——— ④}$$

then eqn ① gives

$$f_0 + \epsilon f_1 + \epsilon^2 f_2 + \text{———} = \int_S (K_0 + \epsilon K_1 + \epsilon^2 K_2 + \cdots)(g_0 + \epsilon g_1 + \epsilon^2 g_2 + \text{———})\, ds$$

$$= \int_S K_0 g_0\, ds + \int_S \epsilon (K_1 g_0 + K_0 g_1)\, ds + \cdots \quad \text{——— ⑤}$$

Comparing coefficient of power of ϵ, we get

$$f_0 = \int_S k_0\, g_0\, ds \quad \text{——} \quad (6)$$

$$f_1 = \int_S (k_1\, g_0 + k_0\, g_1)\, ds \quad \text{——} \quad (7)$$

$$f_2 = \int_S (k_0\, g_2 + k_1\, g_1 + k_2\, g_0)\, ds \quad \text{——} \quad (8)$$

from eq^n (7) $\displaystyle\int_S k_0 g_1\, ds = f_1 - \int_S k_1 g_0\, ds$ —— (9)

similarly from eq^n (8)

$$\int_S k_0\, g_2\, ds = f_2 - \int_S k_1 g_1\, ds - \int_S k_2 g_0\, ds \quad \text{——} \quad (10)$$

This method gives the better result when we impose certain conditions

(i) $k_0(P,Q)$ is the dominating part of $K(P,Q)$

(ii) The eq^n (6) can be solved.

(iii) the function $g_0, g_1 \ldots$ such that the integrals occuring on the R.H.S of eq^n (9) & (10) are easily evaluted. So, in this method the eq^n (6) is needed to be solved.

we shall discuss some different cases

Case-1: Let the kernel K_1 in eq^n (2) is a constant say A. Then from eq^n (1)

$$f(P) = \int_S k_0(P,Q)\, g(Q)\, ds_t + \epsilon \int_S A\, g(Q)\, ds + o(\epsilon^2)$$

$$\Rightarrow f(P) + \epsilon f' = \int_S K_0(P,Q)\, g(Q)\, ds + O(\epsilon^2) \quad \text{--- (11)}$$

where $f' = -\int_S A g(Q)\, ds$

The occurance of a constant 'A' can be demonstrated by the kernel

$$K(P,Q) = \frac{\text{Exp } i\epsilon\, |\vec{r} - \vec{\xi}|}{|\vec{r} - \vec{\xi}|}$$

$$= \frac{\text{Exp } i\epsilon\, \vec{r}}{\vec{r}} = \frac{1}{r}\left[1 + i\epsilon r + \frac{(i\epsilon r)^2}{2!} + \cdots\right]$$

$$= \frac{1}{r} + i\epsilon + \frac{i^2\epsilon^2 r}{2!} + \cdots$$

$$= K_0 + \epsilon K_1 + \epsilon^2 K_2 + \cdots$$

where $K_0 = \frac{1}{r}$, $K_1 = i = A =$ (constant)

$$K(P,Q) = \frac{1}{r} + i\epsilon + O(\epsilon^2) + \cdots$$

Case 2 2018

When the kernel $K(P,Q)$ is only of the form $K_0 + A$

for this case, suppose that the sol^n $G(P)$ of the integral eq^n

$$\int_S K_0(P,Q)\, G(Q)\, ds = 1 \quad \text{--- (12)}$$ is known

and we want to solve

$$\int_S (A + K_0(P,Q))\, g(Q)\, ds = 1 \quad \text{--- (13)}, \quad f(P) = 1$$

We can write eq^n (13) as

$$\int_S K_0(P,Q)\, g(Q)\, ds = 1 - \int_S A g(Q)\, ds \quad \text{—— (14)}$$

or

$$\frac{\int_S K_0(P,Q)\, g(Q)\, ds}{1 - \int_S A g(Q)\, ds} = 1 \quad \text{—— (15)}$$

where $\int_S A g(Q)\, ds$ is constant but yet to be unknown.

Comparing eq^n (15) & (12)

$$G(Q) = \frac{g(Q)}{1 - \int_S A g(Q)\, ds}$$

$$\Rightarrow g(Q) = \left[1 - \int_S A g(Q)\, ds\right] G(Q) \quad \text{—— (16)}$$

Integrating both sides, we get

$$\int_S g(Q)\, ds = \int_S G(Q)\, ds \left[1 - \int_S A g(Q)\, ds\right]$$

$$= \int_S G(Q)\, ds - A\left[\int_S G(Q)\, ds\right] \int_S g(Q)\, ds$$

$$\Rightarrow \int_S \left[1 + A\int_S G(Q)\, ds\right] g(Q)\, ds = \int_S G(Q)\, ds$$

$$\Rightarrow \int_S G(Q)\, ds = \frac{\int_S G(Q)\, ds}{1 + A\int_S G(Q)\, ds} \quad \text{—— (17)}$$

putting eq^n (17) in (16)

$$g(Q) = \left[\frac{1 - A\int_S G(Q)\,ds}{1 + A\int_S G(Q)\,ds} \right] G(Q)$$

$$= \left[\frac{1 + A\cancel{\int_S G(Q)\,ds} - A\cancel{\int_S G(Q)\,ds}}{1 + A\int_S G(Q)\,ds} \right] G(Q)$$

$$g(Q) = \frac{G(Q)}{1 + A\int_S G(Q)\,ds} \quad \text{——— (18)}$$

$$g(P) = \frac{G(P)}{1 + A\int_S G(Q)\,ds} \quad \text{——— (19)}$$

is the required sol^n of eq^n (1) where $G(Q)$ is known as the sol^n of eq^n (12)

2013 (only name) 2016

13-2-20 **Application to electrostatics :-**

Let there be two conductors with surfaces S_1 and S_2. S_1 is completely contained in S_2 and is kept as at a unit potential. where as the potential on S_2 is zero

If 'a' denotes a charactstic length of S_1 and 'b' denotes the minimum distance between a point of S_1 and a point of S_2 then, we have the perservation parameter $\epsilon = \frac{a}{b}$ which we assume to be much

smaller than unity
Now, we presented an integral representation formula for the electrostatic potential in the region D between S_2 and S_1

$$f(P) = \int_{S_1} G(P,Q)\,\sigma(Q)\,ds \quad \text{--- (1)} \quad P \in S_1$$

In terms of green's function $G(P,Q)$ and the charged density σ

Apply the boundary condition on S_1, eq^n (1) becomes:-

$$1 = \int_{S_1} G(P,Q)\,\sigma(Q)\,ds \quad , \quad P \in S_1 \text{ --- (2)}$$

we write $G(P,Q)$ as the sum of free space green's function $G_0(P,Q)$ and the perservation term $G_1(P,Q)$ in eq^n (2), we get

$$1 = \int_{S_1} G_0(P,Q)\,\sigma(Q)\,ds + \int_{S_1} G_1(P,Q)\,\sigma(Q)\,ds \quad , \quad P \in S_1 \text{ --- (3)}$$

If the conductor S_2 were absent we would have only the first integral on ~~write~~ R.H.S of eq^n (3). Thus, the 2nd integral represent the effect of conductor S_2 on the potential of S_1

According to hypothesis, we assume that we can solve integral eq^n ③ when 2nd integral is not present.
Note that we can always introduce a constant A and write $G_1 = A + G_2$
where $G_2 = 0\ (A\epsilon)$
for instance, one possible value of A is the value of $G_1(P,Q)$ for an arbitrary pair of points P and Q on S_1

$$1 = \int_{S_1} G_0(P,Q)\,\sigma(Q)\,ds + A\int_{S_1} \sigma(Q)\,ds + \int_{S_1} G_2(P,Q)\,\sigma(Q)\,ds \quad \text{— ④}$$

Now define a new charge density σ'

$$\sigma'(P) = \frac{\sigma(P)}{1 - A\int_{S_1} \sigma(Q)\,ds} \quad \text{— ⑤}$$

from which it follows that

$$\int_{S_1} \sigma\,ds = \frac{\int_{S_1} \sigma'\,ds}{\left[1 + A\int_{S} \sigma'\,ds\right]} \quad \text{— ⑥}$$

and hence eq^n ④ can be written in terms of density σ' as

$$1 = \int_{S_1} G_0(P,Q)\,\sigma'(Q)\,ds + \int_{S_2} G_2(P,Q)\,\sigma'(Q)\,ds \quad \text{— ⑦}$$

from the proceeding arguments we conclude that second integral on R.H.S of eq^n ⑦ is order of ϵ_{ie}^2 $O(\epsilon^2)$ times the first one.

If we neglect the term of this order then σ' is the electrostatic charge density on S_1, when it is raised to a unit potential in free space

eq^n (6) gives the capacity C of the condensor formed by S_1 and S_2 in terms of free space capacity C_0 of S_1

i.e $$\frac{C}{C_0} = (1 + A C_0)^{-1} + O(\epsilon^2) \quad \text{—— (8)}$$

$[(1+D)^{-1} = 1 - D \ldots]$

or

$$\frac{C}{C_0} = (1 - A C_0) + O(\epsilon^2) \quad \text{—— (9)}$$

If A is interpreted the value of $G_1(P, Q)$ for any pair of points P and Q on S_1 then result (9) is precisely the capacity that would have been obtained had we used the pertervation procedure eq^n (2) through eq^n (4)

The advantage of eq^n (8) for determining the electrostatic capacity lies in the fact that in many situations it is possible to show that by a suitable choice of A the formula (8) is valid for much higher order in ϵ.

14-2-20 **Low - Reynolds - Number Hydrodynamics:**

2018

Two kinds of linearized equations govern the flow of an incompressible viscous fluid:

2015 Stokes & Oseen equation :-

(1) **Steady Stokes flow** :-

For a free space the boundary value problem is

$$\nabla^2 \vec{q} = \text{grad}\, p, \quad \text{div}\vec{q} = 0 \quad \text{——} ①$$

$$\vec{q} = \hat{e}_1 \text{ on } S_1 ; \quad q_1 \to 0 \text{ at } \infty \quad \text{——} ②$$

where this system has been made dimensionless with the help of uniform speed U of the solid and with its characteristic length 'a'
Here $\hat{e}_1$ is the unit vector along x_1-axis.
The integral eq^n formula for this B.V.P was found in terms of green's tensor T_1 and green's vector $\vec{p}_1$ to be

$$\hat{e}_1 = -\int_{S_1} \vec{f}\cdot T_1 ds, \quad p \in S_1 \quad \text{——} ③$$

$$\text{where } \vec{f} = \left(\frac{\partial \vec{q}}{\partial n}\right) - p\hat{n} \quad \text{——} ④$$

$$T_1 = \frac{1}{8\pi}\left[\vec{I}\,\nabla^2 |\vec{x} - \vec{\xi}_1| - \text{grad grad } |\vec{x} - \vec{\xi}_1|\right) \quad \text{——} ⑤$$

$$\vec{p}_1 = -\frac{1}{8\pi}\left(\text{grad } \nabla^2 |\vec{x} - \vec{\xi}_1|\right) \quad \text{——} ⑥$$

The corresponding formula for the resistance $\vec{F}_\infty$. The subscript signifies that we have an infinite mass of fluid on the body

B is found by observing that stress tensor has the value

$$\vec{I}p + [\nabla\vec{q} + (\nabla\vec{q})^t]$$

where $(\nabla\vec{q})^t$ stands for transpose of $\nabla\vec{q}$ using eq^n (4), we have

$$\vec{F}_\infty = \int_{S_1} \vec{f}_\infty \, ds \quad \text{—— (7)}$$

This force can be related to so-called resistance tensor

$\vec{\Phi}_\infty$ which is defined to be such that the force exerted on a body with uniform velocity $\vec{u}$ is $\Phi_\infty \cdot \vec{u}$

Thus $\vec{F}_\infty = F_\infty \hat{e} = -\vec{\Phi}_\infty \vec{u}$
where $\hat{e}$ is the unit vector in the direction of $\vec{u}$. The sol^n for various B.V.P for steady stoke's flow in an unbounded medium are known as such the sol^n of integral eq^n (3) can be found for these problems.

Hence the tensor T_1 corresponds to kernel K_0.

2015
(2) <u>Boundary effects on stoke's flow</u>:

The presence of the boundary S_2 the necessity. The introduction of a new tensor '$\vec{T}$' and a corresponding vector $\vec{t}$. These quantities satisfy the eq^n

$$\left.\begin{aligned} \nabla^2 \vec{T} - \text{grad}\, \vec{p} &= \vec{I}\, \delta(\vec{x} - \vec{\xi_1}) \\ \nabla \cdot \vec{T} = 0, \quad T &= 0 \text{ on } S_2 \end{aligned}\right\} \quad (8)$$

when $S_2 \to \infty$, $\vec{T}$ & $\vec{p}$ reduces to $\vec{T_1}$ & $\vec{p_1}$

According to present scheme, we write

$$\vec{T} = \vec{T_1} + \vec{T_2} \quad \& \quad \vec{p} = \vec{p_1} + \vec{p_2}$$

where $\vec{T_2}$ & $\vec{p_2}$ satisfies the homogenous part of the system (8)

The integral eq^n i.e equivalent to present problem is

$$\hat{e_1} = -\int_{S_1} \vec{f} \cdot \vec{T}\, ds = -\int_{S_1} \vec{f} \cdot (\vec{T_1} + \vec{T_2})\, ds \quad (9)$$

Along with P and Q. Let us take the origin also on S_1. Let ϵ be the parameter that gives the ratio of 'a', the standard geometric length of the solid B, to the minimum distance b/w a point of S_1 and a point of S_2

Then by the taylor's thm, we get

$$\vec{T_2} = \vec{T_2}^{\circ} + x\left[\text{grad}\, \vec{T_w}\right]_{x=\xi_1=0} + \xi_1 \left[\text{grad}^{\circ}\, T_2\right]_{x=\xi_1=0} + o(\epsilon^3) \quad (10)$$

where $\vec{T_2}^{\circ} = \vec{T_2}(0,0)$

and the superscript zero on the gradient implies diffentiation w.r.t the components of $\vec{e}_1$.

Taking only the first order terms of the relation (10) in (9), there results the eq^n

$$\hat{e}_1 + \vec{F}\cdot\vec{T}_2^{\,0} = -\int_{S_1} \vec{f}\,\vec{T}_1\,ds \quad \text{--- (11)}$$

where $\vec{F}$ is defined by (7) without the subscript infinity in the relation

i.e F is the resistance experiened by B in the bounded medium.

The integral eq^n (11) has same kernel as that of eq^n (3) and as such it can be considered to give the velocity field in an unbounded fluid. When B is moving with uniform velocity

$$\hat{e}_1 + \vec{F}\cdot T_2^{\,0}$$

If we now utilize the concept of resistance Φ_∞ as previously define.

we derive the force formula

$$\vec{F} = -(\hat{e}_1 + \vec{F}\cdot\vec{T}_2^{\,0})\,\vec{\Phi}_\infty \quad \text{--- (12)}$$

This eq^n can be solved to give

$$\vec{F} = -\hat{e}_1\left[\vec{\Phi}_\infty^{-1} + \vec{T}_2^{\,0}\right]^{-1} \quad \text{--- (13)}$$

Replacing $\vec{F}$ by $\vec{F_\infty}$ introduce error of order ϵ^2 and thus to order ϵ, the formula (12) becomes

$$\vec{F} = -\left[\hat{e}_1 + \vec{F_\infty} \cdot \vec{T_2}^{\circ}\right] \Phi_\infty \quad \text{——} \quad (14)$$

The principle axis of the resistance of B are defined so that when B moves parallel to one of them in an infinite mass of fluid, the force is in direction of motion

They are the unit Eigen vectors of the resistance tensor $\vec{\Phi}_\infty$. Let us denote them $\hat{i}_1, \hat{i}_2$ & $\hat{i}_3$ such that $\vec{\Phi}_\infty = \vec{\Phi}\, \hat{i}_1 \hat{i}_1 + \vec{\Phi}_{\infty 2} \hat{i}_2 \hat{i}_2 + \vec{\Phi}_\infty \hat{i}_3 \hat{i}_3$

—— (15)

Let us decompose $\vec{T_2}^{\circ}$ into components with these eigen vectors as the basis

Let us see $\hat{e}_1 = \hat{i}_1$ substituting these expression in relation (12), we derive

$$\frac{\vec{F}}{\vec{F_\infty}} = \frac{1}{1 - \lambda \vec{F_\infty}} \quad \text{——} \quad (16)$$

where λ is independent of the form of S_1

1-2-20

Longitudinal oscillations of solids in Stokes flow:

This analysis can be used to obtain an approximate value of the velocity field generated and the resistance experienced by a solid of an arbitrary shape which is executing slow longitudinal vibrations in an unbounded viscous fluid. Let us assume that the body oscillates about some mean position with velocity $Ue^{i\omega t}e_1$ and q and p have the same time dependence. Then, the dimensionless Stokes equations for the steady-state vibrations are

$$-\nabla p + \nabla^2 \vec{q} - iM^2 \vec{q} = 0, \quad \text{div}\,\vec{q} = 0 \quad \text{---} (17)$$

where $M^2 = a^2\omega/\nu$ is the rotational Reynolds number and ν is the coefficient of kinematic viscosity.

The integral representation formulas are the same as for the steady stokes flow, and T and p now satisfy the equations

$$-\nabla \vec{p} + \nabla^2 \vec{T} - iM^2 \vec{T} = \vec{I}\,\delta(x - \xi), \quad \text{div}\,\vec{T} = 0 \quad \text{---} (18)$$

and $T \to 0$ as $x \to \infty$. These equations are satisfied when T and p are given by the formulas

$$\vec{T} = \vec{I}\,\nabla^2\phi - \text{grad grad}\,\phi, \quad \vec{p} = -\text{grad}(\nabla^2 - iM^2)\phi, \quad \text{---} (19)$$

$$(\nabla^2 - iM)\nabla^2\phi = \delta(\vec{x} - \vec{\xi}) \quad \text{---} (20)$$

$$\phi = \frac{1 - \exp\{-[(1+i)M/\sqrt{2}]\,|x - \xi|\}}{4\pi i M^2 |x - \xi|} \quad \text{---} (21)$$

Thus,

$$\vec{T} = T_1 - [(1+i)/6\pi\sqrt{2}]\,M\vec{J} + O(M^2) \quad \text{---} (22)$$

where $\vec{T}_1$ is given by eq^n (5)

The next step is to substituate the boundary value $q = e_1$ in the integral representation formula for the system (17) and observe that, in view of eq^n (18) and green's theorem, we have

$$\iint_{S_1} \left(\frac{\partial \vec{T}}{\partial n} - pn \right) ds = -iM^2 \int_{R_1} T \, dv \quad \text{—— (23)}$$

where R_1 is the interior of S_1. The result is the fredhlom integral eq^n of the first kind for evaluting f

$$e_1 - [(1+i)/6\pi\sqrt{2}] MF = -\int_{S_1} T_1 . f \, dS + o(M^2) \quad , P \in S_1 \quad \text{—— (24)}$$

following the previous analysis, we have the formula:

$$\vec{F} = \vec{\Phi}_\infty \cdot [e_1 - [(1+i)/6\pi\sqrt{2}] M [\vec{\Phi}_\infty e_1)] + o(M^2) \quad \text{—— (25)}$$

for a body moving parallel to one of its axes of resistance (which we can take as the x_1-axis of our coordinate system), eq^n (25) take the simple form.

$$\vec{F} = -F_\infty \{ 1 + [(1+i)/6\pi\sqrt{2}] M F_\infty \} e_1 \quad \text{—— (26)}$$

for example, for a sphere, $F_\infty = 6\pi\mu a U$ is physical units, where a is the radius of the sphere and μ is the shear viscosity of the fluid.

The formula (26) then gives (in physical units)

$$\vec{F} = -6\pi\mu a U \left[1 + (M/\sqrt{2})(1+i)\right]\vec{e}_1 + O(M^2) \quad \text{--- (27)}$$

2014

Steady rotary Stokes flow:

For the rotation of axially symmetric bodies, the pressure is taken to be constant and the steady Stokes equations becomes

$$\nabla^2 \vec{q} = 0 \;,\quad \operatorname{div}\vec{q} = 0 \qquad p = \text{Constant} \quad \text{--- (28)}$$

Let the z-axis of cylindrical polar coordinates (ρ, ϕ, z) be the axis of symmetry of these bodies. Assuming that the streamlines are circles lying in planes perpendicular to Oz, then $\vec{q}$ has a nonzero component $v(\rho, z)$ in the ϕ direction only and is independent of ϕ. The eq^n of continuity is thus satisfied automatically, and the eq^n of motion (28) becomes

$$\left(\frac{\partial^2 v}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial v}{\partial \rho} + \frac{\partial^2 v}{\partial z^2} - \frac{v}{\rho^2}\right) = 0$$

$$\text{i.e} \left(\frac{\partial^2 v}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial v}{\partial \rho} + \frac{\partial^2 v}{\partial z^2} - \frac{v}{\rho^2}\right) = 0 \quad \text{--- (29)}$$

which has been made dimenionless with Ωa as the typical velocity. Here, Ω is the uniform angular velocity of the body and a is its characteristic length. The boundary conditions are

$$v = \rho \text{ on } S_1 \;;\quad v = 0 \text{ on } S_2 \quad \text{--- (30)}$$

from eq^n (29). It is easily verified that the function

$$w(\rho, \psi, z) = v(\rho, z)\cos\psi \quad \text{——— (31)}$$

is harmonic and can, therefore be represented in terms of a source density $\sigma(Q)\cos\psi_1$ spread over S_1, where $\xi = (\rho_1, \psi_1, z)$ are the coordinates of Q and $\sigma(Q)$ is independent of ψ_1. Thus, we can use the integral representation formula

$$\left(f(x) = \int_S G(\vec{x}, \vec{\xi})\sigma(\vec{\xi})\,ds \right) \text{ for the}$$

harmonic function and get

$$w(\rho, \psi, z) = \int_{S_1} G(P, Q)\,\sigma(Q)\cos\psi_1\,ds \quad \text{——— (32)}$$

where P is an arbitrary point in the region between S_1 and S_2. On applying the boundary condition (30), we obtain

$$\rho = \int_C \rho_1 G^{(1)}(P, Q)\,\sigma(Q)\,ds \qquad P \in S_1 \quad \text{——— (33)}$$

where $\pi^{-1} G^{(1)}(P, Q)$ is the coefficient of $\cos(\psi - \psi_1)$ in the fourier expansion of $G(P, Q)$ and ds denotes the element of the arc length measured along the curve C which is the bounding curve of S_1 in the meridian plane.

Recall the decomposition

$$G(P, Q) = (1/|\vec{x} - \vec{\xi}|) + G_1(P, Q)$$

where $G_1(P, Q)$ is finite in the limit as $Q \to P$. We can, similarly, decompose the fourier componant $G^{(1)}$ into the sum

$$G^{(1)} = G_0^{(1)} + G_1^{(1)}.$$

where $G_0^{(1)}$ arises from the fourier expansion of $1/|\vec{\alpha} - \vec{\xi}_1|$ and $G_1^{(1)}$ arises from the expansion of G_1. Therefore, we can write eq^n 33 as

$$\rho = \int_C \rho_1 G_0^{(1)} \sigma \, ds + \int_C \rho_1 G_1^{(1)} \sigma \, ds \quad \text{—— (34)}$$

Again let b represent the minimum distance between a point of S_1 and a point of S_2, and we have the small perturbation parameter $\epsilon = a/b$. The second integral on the right side of eq^n 34 is at least of order ϵ of the first integral. For geometric configurations for which

$$G_1^{(1)} = \rho \rho_1 (A + G_2) \quad \text{—— (35)}$$

where A is a constant and G_2 is of order $A\epsilon$, eq^n 34 becomes

$$\rho = \int_C \rho_1 G_0^{(1)} \sigma \, ds + A\rho \int_C \rho_1^2 \sigma \, ds + \rho \int_C \rho_1^2 G_2 \sigma \, ds \quad \text{—— (36)}$$

or

$$\rho = \int_C \rho_1 G_0^{(1)} \sigma \, ds + \rho \int_C \rho_1^2 G_2 \sigma' \, ds \quad \text{—— (37)}$$

Where $\sigma' = \sigma / \left(1 - A \int_C \rho_1^2 \sigma \, ds\right)$ —— (38)

Consequently, we have the same situation as in the section on electrostatics, that is, σ' represents with an error which is at most of order ϵ^2, an appropriate source density for the body rotating in an infinite mass of fluid.

The tangential stress component τ on the surface S_1 in the direction of ψ increasing is

$$\tau = \mu \rho \frac{\partial}{\partial n}\left(\frac{v}{\rho}\right) \quad \text{——— (39)}$$

where $\frac{\partial}{\partial n}$ denotes differentiation along the normal drawn outward to S_1.

Furthermore, we know that the source density $\sigma(Q)$ on S_1 is related to v by

$$4\pi\sigma(Q) = -\rho\frac{\partial}{\partial n}\left(\frac{v}{\rho}\right) \quad \text{——— (40)}$$

Thus $\tau = -4\pi\mu\sigma$. From this value of the stress component, the value of the frictional torque N can now be readily calculated to be

$$N = -8\pi^2\mu\int_C \rho^2\sigma\, ds \quad \text{——— (41)}$$

The relation between this Torque N and the torque N_∞ is an unbounded fluid may be obtained by integrating both sides of the relation (38) around the meridian section C of the axially symmetric body:

$$N = N_\infty \left[1 + (A/8\pi^2 \mu \Omega) N_\infty\right]^{-1} \quad \text{--- (42)}$$

with an error of order ϵ^2. By a suitable choice of A, the formula (42) can be shown to be valid in many cases to a much higher order in ϵ. Eq^n (42) can be illustrated with many interesting configurations for example, the case of a sphere which is symmetrically placed in an infinite cylindrical shell formula (42) then gives

$$N/N_\infty = \left[1 + (N_\infty / 8\pi \mu \Omega a^3) H_1\right] \quad \text{--- (43)}$$

Where H_1 is given by the integral

$$H_1 = \frac{2}{\pi} \frac{(-1)^k}{(2k)!} \int_0^\infty x^{2k} \frac{K_1(x)}{I_1(x)} \, dx.$$

2017

Rotary oscillations in Stokes flow:

The equations governing the steady state rotary oscillations (with circular frequency ω) of axially symmetric solids in an incompressible viscous fluid are

$$(\nabla^2 - iM^2) q = 0 \quad , \quad \nabla \cdot q = 0 \quad \text{--- (44)}$$

which are obtained from eq^n (17) by setting ρ = constant. As for the preceding steady rotational case, the only non-zero component of q is the ψ component v, and the differential eq^n (44) reduce to solving the eq^n

$$\frac{\partial^2 v}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial v}{\partial \rho} - \frac{v}{\rho^2} + \frac{\partial^2 v}{\partial z^2} - \beta^2 v = 0 \quad \text{--- (45)}$$

where $\beta^2 = iM^2$. we present the analysis for $\beta \ll 1$. The boundary values are

$$v = \rho \text{ on } S_1 \; ; \; v = 0 \text{ on } S_2 \quad \text{--- (46)}$$

where, as before, S_1 is the surface of the oscillating body and S_2 is the bounding surface.

By writing $w = v \cos\psi$, eq^n (45) & (46) reduce to the following boundary value problem.

$$(\nabla^2 - \beta^2) w = 0 \quad \text{--- (47)}$$

$$w = \rho\cos\psi \text{ on } S_1 \; ; \; w = 0 \text{ on } S_2 \quad \text{--- (48)}$$

The Green's function $G(\vec{x}; \vec{\xi})$ appropriate to this boundary value problem is

$$(\nabla^2 - \beta^2)\, G(\vec{x}; \vec{\xi}) = -4\pi\, \delta(\vec{x} - \vec{\xi}) \quad , \quad G|_{S_2} = 0 \quad \text{--- (49)}$$

Thus,
$$G(\vec{x}; \vec{\xi}) = \frac{\exp - \beta|\vec{x} - \vec{\xi}|}{|\vec{x} - \vec{\xi}|} + G_1(\vec{x}, \vec{\xi}) \quad \text{--- (50)}$$

where $G_1(\vec{x}, \vec{\xi})$ is finite in the limit as $\vec{\xi} \to \vec{x}$. The integral representation formula for $w(x)$

$$w(x) = \int_{S_1} \sigma(\rho_1, z_1)(\cos\psi_1)\, G(\vec{x}, \vec{\xi})\, ds, \quad \vec{\xi} \in S_1, \; x \in R \quad \text{--- (51)}$$

where R is the region between S_1 and S_2, and $\sigma(\rho_1, z_1)$ is given by formula (40) when we apply the boundary condition (48), we obtain the required fredhlom integral eq^n

$$\rho \cos\psi = \int_{S_1} \sigma(\rho_1, z_1)(\cos\psi_1)\, G(\vec{r}, \vec{\xi})\, ds \quad \text{—— (52)}$$

with r and ξ on S_1. Now, if $G_1^{(1)}(\rho, z; \rho_1, z_1)$ is the coefficient of $\cos(\psi - \psi_1)$ in the fourier expansion of $G_1(b r; \xi)$, then the integration over ψ_1 reduces the preceding integral to

$$\rho \cos\psi = \int_{S_1} \sigma(\rho_1, z_1)(\cos\psi_1) \frac{\exp -\beta|\vec{r} - \vec{\xi}|}{|\vec{r} - \vec{\xi}|}\, ds$$

$$+ \pi(\cos\psi) \int_{C} \sigma(\rho_1, z_1)\, G_1^{(1)}(\rho, z; \rho_1, z_1)\, \rho_1\, ds \quad \text{—— (53)}$$

in the notation of eq^n (33)
The next step is to expand σ as the perturbation series

$$\sigma = \sum_n \beta^n \sigma_n \quad \text{—— (54)}$$

In eq^n (53). Moreover, by direct expansion of the Green's function, it can be shown that $G_1^{(1)} = O(\epsilon^3)$, where ϵ is the ratio of the characteristic length of the vibrating body to the distance of its center from the nearest point of S_2. It is assumed that $q = \beta/\epsilon = O(1)$

Now, equate equal powers of β on both sides of eq^n (53) and get (after omitting terms that trivially vanish)

$$\rho \cos\psi = \int_{S_1} \sigma_0(\rho_1, z_1)\, |\vec{\mu} - \vec{\xi}_1|^{-1} \cos\psi_1\, ds \qquad (55)$$

$$0 = \int_{S_1} \sigma(\rho_1, z_1)\, |\vec{\mu} - \vec{\xi}_1|^{-1} \cos\psi_1\, ds \qquad (56)$$

$$0 = \int_{S_1} \sigma_1(\rho_1, z_1)\, |\vec{\mu} - \vec{\xi}_1|^{-1} \cos\psi_1\, ds + \frac{1}{2}\int_{S_1} \sigma_0(\rho_1, z_1)\, |\vec{\mu} - \vec{\xi}_1| \cos\psi_1\, ds \qquad (57)$$

$$0 = \int_{S_1} \sigma_3(\rho_1, z_1)\, |\vec{\mu} - \vec{\xi}_1|^{-1} \cos\psi\, ds + \frac{1}{2}\int \sigma_1(\rho_1, z_1) |\vec{\mu} - \vec{\xi}_1| \cos\psi\, ds - \frac{1}{6}\int_{S_1} \sigma_0(\rho_1, z_1)\, |\vec{\mu} - \vec{\xi}_1|^2 \cos\psi\, ds$$

$$+ \frac{1}{\pi}(\cos\psi) \int_C \sigma_0(\rho_1, z_1)\, H(\rho, z; \rho_1, z_1)\, \rho_1\, ds \qquad (58)$$

and so on, where

$$G_1^{(1)}(\rho, z; \rho_1, z_1) = \beta^2 H(\rho, z; \rho_1, z_1) + O(\beta^4) \qquad (59)$$

It follows from eqn's (55) through (58) that the source densities $\sigma_0, \sigma_1, \sigma_2, \sigma_3$ and so on are determined by solving potential problems in free space.

The velocity field and the frictional torque can be readily calculated. Indeed, for the evaluation of the torque N, we use the formula (39) and obtain

$$N = \mu \int_S \rho^2 \frac{\partial}{\partial n}\left(\frac{\nu}{\rho}\right) ds = 2\pi\mu \int_C \rho^3 \frac{\partial}{\partial n}\left(\frac{\nu}{\rho}\right) ds \quad \text{--- (60)}$$

from the relations (40), (54) and (60), it follows that

$$N = -8\pi^2\mu \int_C \rho^2 (\sigma_0 + \beta\sigma_1 + \beta^2\sigma_2 + \beta^3\sigma_3)\, ds + O(\beta^4) \quad \text{--- (61)}$$

Since potential problems of the type given in eqn's (55) through (58) can be solved for various configurations such as a sphere, a spheroid, a lens, and a thin circular disk, we can solve our problem for all these geometric shapes. As an example; we consider the case of a thin circular disk vibrating about its axis in a viscous fluid which is contained in an infinite circular cylinder. The axes of the disk and the cylinder coincide. The Green's function for an infinite cylinder $-\infty < z < \infty$, $0 \leq \rho \leq b$.

The result is

$$G(\vec{r};\vec{\xi}) = \frac{\exp -\beta|\vec{r}-\vec{\xi}|}{|\vec{r}-\vec{\xi}|} - \frac{2}{\pi}\sum_{n=0}^{\infty}(2-\delta_{0n})\left[\cos n(\psi-\psi_1)\right]$$
$$\times \int_\beta^\infty \frac{K_n(pb)\, I_n(p\rho)\, I_n(p\rho_1)}{I_n(pb)} \left\{\cos\left(p^2-\beta^2\right)(z-z_1)\right\}$$
$$\times \frac{p\,dp}{(p^2-\beta^2)^{1/2}} \quad \text{--- (62)}$$

from which $G_1^{(1)}$ for the disk $\rho \leq 1$, $0 \leq \psi \leq 2\pi$, $z=0$ may be readily obtained:

$$G_1^{(1)}(\rho,\rho_1) = -\frac{1}{2\pi}\epsilon^3 \rho\rho_1 \int_q^\infty \frac{K_1(y)\, y^3\, dy}{I_1(y)\,(y^2-q^2)^{1/2}} + o(\epsilon^5) \qquad (63)$$

or

$$H(\rho,\rho_1) = -(1/2\pi q^3)\,\rho\rho_1 A(q) \qquad (64)$$

Here $A(q)$ stands for the infinite integral in eq^n (63)

The integral eq^n (55) through (58)

The solutions are

$$\left.\begin{aligned}
\sigma_0 &= \frac{2\rho}{\pi^2(1-\rho^2)^{1/2}}, \qquad \sigma_1 = 0 \\
\sigma_2 &= \frac{\rho(2-\rho^2)}{3\pi^2(1-\rho^2)^{1/2}} \\
\sigma_3 &= \frac{4}{3\pi^2}\left[\frac{-2}{3} + \frac{1}{\pi q^3}A(q)\right]\frac{\rho}{(1-\rho^2)^{1/2}}, \quad 0 \le \rho \le 1
\end{aligned}\right\} \qquad (65)$$

Substituting these values in eq^n (61), we obtain the value of the torque, which is physical units is.

$$N = -\frac{32}{3}\mu\Omega a^3\left[1 + \frac{1}{5}\beta^2 - \frac{4}{9\pi}\beta^3 + \frac{4}{3\pi^2}\epsilon^3 A(q)\right]e^{i\omega t} + o(\beta^4, \epsilon^5) \qquad (66)$$

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Oseen flow - translational motion:-

The slow motion past a solid as studied by Oseen is governed by the dimensionless equations

$$R\,\partial q/\partial x = -\text{grad}\,p + \nabla^2 q \quad , \quad \text{div}\,q = 0 \quad \text{—— (67)}$$

$$q = e_1 \text{ on } S_1 \; ; \; q = 0 \text{ on } S_2 \quad \text{—— (68)}$$

The fredhlom integral eq^n of the first kind that is equivalent to the boundary value problem (67) through (68)

$$e_1 = -\int_S T.f\,ds \quad \text{—— (69)}$$

where the Green's tensor T and the Green's vector P are now defined as

$$\left. \begin{aligned} T &= (1/8\pi)\left[I\nabla^2\phi - \text{grad grad}\,\phi\right] \\ P &= -(1/8\pi)\,\text{grad}\,(\nabla^2\phi - R\,\partial\phi/\partial x_1) \end{aligned} \right\} \text{—— (70)}$$

$$\phi = (1/|\sigma|)\int_0^{|\sigma| S} \left[(1-e^{-t})/t\right]dt$$

and $S = |\vec{x} - \vec{\xi}| + (R/|R|)(x_1 - \xi_1)$

By using the series

$$\frac{1-e^{-t}}{t} = 1 - \frac{t}{2!} + \frac{t^2}{3!} + \cdots$$

we expand ϕ is eq^n (71) in terms of the Raynolds number. The relation (70), then becomes

$$T = T_1 + O(R_e)$$

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Oseen flow - rotary motion:

By using the present technique, the solutions of the oseen equations can be presented also for the steady rotations of axially symmetric solids. As in the corresponding stokes flow case, we take p = constant. Then, the oseen equations take the simple form

$$R_e(\partial q/\partial x_1) - \nabla^2 q = 0, \quad \text{div} q = 0 \quad \text{——} (73)$$

Again, in view of the symmetry, only the ϕ component V of q is non-zero and in cylinderical polar coordinates (with $z = x_1$) the boundary value problem becomes

$$\frac{\partial^2 V}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial V}{\partial \rho} - \frac{V}{\rho^2} + \frac{\partial^2 V}{\partial z^2} - 2c\frac{\partial V}{\partial z} = 0 \quad \text{——} (74)$$

where $c = Ua/2\nu = R_e/2$. The boundary conditions on V are

$$V = \rho \text{ on } S_1 \; ; \; V = 0 \text{ on } S_2 \quad \text{——} (75)$$

The substitution of $V = e^{cz}\,\upsilon(\rho, z)$ reduces this boundary value problem to the following one.

$$\frac{\partial^2 \upsilon}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \upsilon}{\partial \rho} - \frac{\upsilon}{\rho^2} + \frac{\partial^2 \upsilon}{\partial z^2} - c^2\upsilon = 0 \quad \text{——} (76)$$

$$v = \rho e^{cz} \quad \text{on } S_1 \; ; \quad v = 0 \quad \text{on } S_2 \qquad \text{—— (77)}$$

equation (76) is the same as (45) with β replaced by c. However, the boundary conditions (77) and (46) are different. By repeating the algebraic steps (47) through (53), we end up with the fredhlom integral eq^n

$$\rho e^{cz} \cos\psi = \int_{S_1} \sigma(\rho_1, z_1) \, (\cos\psi_1) \left[(\exp - c\,|\vec{\mu} - \vec{\xi}_1|) / |\vec{\mu} - \vec{\xi}_1| \right] ds$$

$$+ \pi (\cos\psi) \int_C \sigma(\rho_1, z_1) \, G_1^{(1)} (\rho, z; \rho_1, z_1) \rho_1 \, ds$$

—— (78)

for the evaluation of (ρ, z) defined by the relation (40). Although the only difference between the integral eq^n's (53) & (78) is in the expression on their left side, it leads to a much more difficult analysis for the present problem. To solve eq^n (78), we again take the expansion $\sigma = \sum_n c^n \sigma_n$ in (54) and also expand

$\rho e^{cz} \cos\psi$ in power series of c. By comparing the equal powers of c in (78), we obtain the following integral eq^n's of potential theory:

$$\rho \cos\psi = \int_{S_1} \sigma_0(\rho_1, z_1) \, |\vec{\mu} - \vec{\xi}_1|^{-1} \cos\psi_1 \, ds \qquad \text{—— (79)}$$

$$\rho z \cos\psi = \int_{S_1} \sigma_0(\rho_1, z_1) \, |\vec{\mu} - \vec{\xi}_1|^{-1} \cos\psi_1 \, ds \qquad \text{—— (80)}$$

$$\frac{1}{2}\rho z^2 \cos\psi = \int_{S_1} \sigma_2(\rho_1, z_1)\, |\vec{x} - \vec{\xi}_1|^{-1} \cos\psi_1\, ds + \frac{1}{2}\int_{S_1} \sigma_0(\rho_1, z_1)\, |\vec{x} - \vec{\xi}_1| \cos \cos\psi_1\, ds \quad (81)$$

$$\frac{1}{6}\rho z^3 \cos\psi = \int_{S_1} \sigma_3(\rho_1, z_1)\, |\vec{x} - \vec{\xi}_1|^{-1} \cos\psi_1\, ds + \frac{1}{2}\int_{S_1} \sigma_1(\rho_1, z_1)\, |\vec{x} - \vec{\xi}_1| \cos \cos\psi_1\, ds - \frac{1}{6}\int_{S_1} \sigma_0(\rho_1, z_1)\, |\vec{x} - \vec{\xi}_1|^2 \cos\psi_1\, ds$$

$$+ \pi(\cos\psi)\int_{G} \sigma_0(\rho_1, z_1)\, H(\rho, z; \rho_1, z_1)\, ds \quad (82)$$

where $H(\rho, z; \rho_1, z_1)$ is defined by relation (59)

for a thin circular disk $z = 0$, $\rho \leq 1$, the system of equations (79) through (82) is the same as the system (55) through (58). Thus, the sol^n for the steady rotation problem for the disk in Oseen flow is the same as the corresponding sol^n for the steady-state vibrations in Stokes flow. for example, the value of the torque N in the present case can be deduced from the formula (66)

$$N = -\frac{32}{3}\mu\Omega a^3 \left\{ 1 + \frac{c^2}{5} - \frac{4c^3}{9\pi} + \frac{4}{3\pi^2}\epsilon^3 A(q) \right\} + O(c^4, \epsilon^5) \quad (83)$$

where Ω is the uniform angular velocity of the solid.

for other configuration, one has to solve the integral equations (79) through (82) with a non-zero left side. We illustrate this by considering the rotation of a sphere of radius a. In this case, it is convenient to take spherical polar coordinates (r, ϑ, ψ). The Value of Green's function $G(\vec{x}; \vec{\xi}_1)$ is the same as (62) with β replaced by c. The corresponding Values of $G_1^{(1)}$ and $H(\vartheta, \vartheta_1)$ are

$$G_1^{(1)}(\vartheta, \vartheta_1) = \frac{-1}{2\pi} \epsilon^3 (\sin\vartheta \sin\vartheta_1) \int_1^\infty \frac{K_1(y)\, y^3\, dy}{I_1(y)(y^2 - q^2)^{1/2}} + O(\epsilon^5) \quad (84)$$

and

$$H(\vartheta, \vartheta_1) = \frac{-1}{2\pi q^3} (\sin\vartheta \sin\vartheta_1) A(q) \quad (85)$$

The source densities $\sigma_0, \sigma_1, \sigma_2$ and σ_3 are determined from equations (79) through (82).

The result is

$$\left.\begin{aligned}
\sigma_0 &= (3/4\pi)\, P_1'(\cos\vartheta) \\
\sigma_1 &= (5/12\pi)\, P_2'(\cos\vartheta) \\
\sigma_2 &= (1/4\pi)\left[(3/2)(P_1'(\cos\vartheta) + (7/15)\, P_3'(\cos\vartheta)\right]
\end{aligned}\right\}$$

$$\sigma_3 = -(3/4\pi)\, P_1'(\cos\vartheta)\left[(1/3) + (1/2\pi q^3) + (1/2\pi q^3) A(q)\right] \quad (86)$$

$0 < \vartheta < \pi$. substituting these values in the torque formula (61), we obtain (in physical units)

$$N = -8\pi \mu \Omega a^3 \left[1 + \frac{4c^2}{15} - \frac{c^3}{3} + \frac{1}{2\pi} \epsilon^3 A(q) \right] + O(c^4, \epsilon^5) \quad (87)$$