Diff. Equation.
CLASS – M.Sc (P) Maths.
Unit – 3 (Pdf. 3).



## Limit Set of Half path :-

**Definition :-** Let $C^+$ be a half-path of system ① defined by $x = f(t)$, $y = g(t)$ for $t \geq t_0$.
Let $(x_1, y_1)$ be a point in the xy-Plane. If there exists a Sequence of real numbers $\{t_n\}$, $n = 1, 2, 3, \ldots$ such that $t_n \to +\infty$ and $[f(t_n), g(t_n)] \to (x_1, y_1)$ as $n \to \infty$, then we call $(x_1, y_1)$ a limit point of $C^+$.
The set of all limit points of a half path $C^+$ will be called the limit set of $C^+$ and denoted by $L(C^+)$.

⊛ **Poincare Bendixon Thm :-** Let $C^+$ be a +ive semi-orbit (half-path) of the System ① contained in a Closed subset K of domain D and $L(C^+)$ consists of regular point only (No critical points) of the given system ①. Then either $C^+$ is a periodic orbit or $L(C^+)$ is a Periodic orbit.

**Proof :-** If $C^+$ is a Periodic orbit, (Closed path), then in the neighbourhood of any point of $C^+$ contains points of $C^+$ other than that point of $C^+$.
$\Rightarrow$ Every point of $C^+$ is a limit point of $C^+$.
$\therefore$ $C^+ = L(C^+)$ [Set of all limit points of Half path $C^+$]
$\Rightarrow$ $L(C^+)$ is a Periodic orbit.

Now, let us suppose that $C^+$ is NOT a periodic orbit.
Then we have to prove that $L(C^+)$ is a periodic orbit.
Since $L(C^+)$ is a non-empty set and contains regular points only (No critical points in $L(C^+)$)

By using the thm :- let $C^+$ be a +ive semi orbit contain in a closed subset $k$ of domain $D$ and assuming that $L(C^+)$ consist of regular points only, then the orbit $C_0$ through its regular point $0$ exists as full orbit and $C_0 \subseteq L(C^+)$.

$\therefore$ we can says that $\exists$ a limit orbit $C_0$ in $L(C^+)$.

$\therefore$ $C_0 \subseteq k$ & $C_0^+$ be the +ive semi-orbit of $C_0$.

$\therefore C_0^+ \subseteq k$, Now being the limit orbit $C_0^+$ has a limit point $P_0$ of $C_0^+$ & hence of $L(C^+)$.

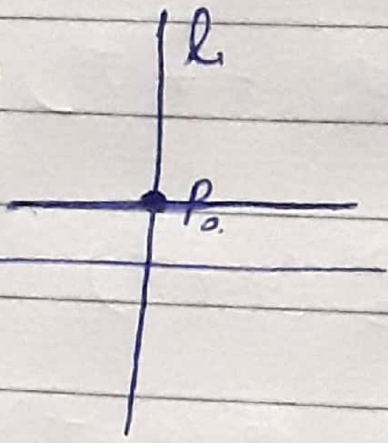
{ $C_0^+$ is closed set and closed set contains all its limit point inside it. }

But $L(C^+)$ is closed

$\therefore$ $P_0 \in L(C^+)$.

Let '$l$' be the transversal through this point $P_0$.

$P_0$ and $C_0^+$ are both in $L(C^+)$.




By using the Result ; "If $C^+$ and $L(C^+)$ have a point in common, then $C^+$ is a periodic orbit."

The Transversal line '$l$' cannot meet $L(C^+)$ at more than one point & that point is $P_0$. Since $P_0$ is a limit point of $C_0^+$.

$\therefore$ $P_0$ is a common point for $C_0^+$ & $L(C^+)$

$\Rightarrow$ $C_0^+$ is a periodic orbit [By above Result]

Hly, we can show that $C_0^-$ is also a periodic orbit.

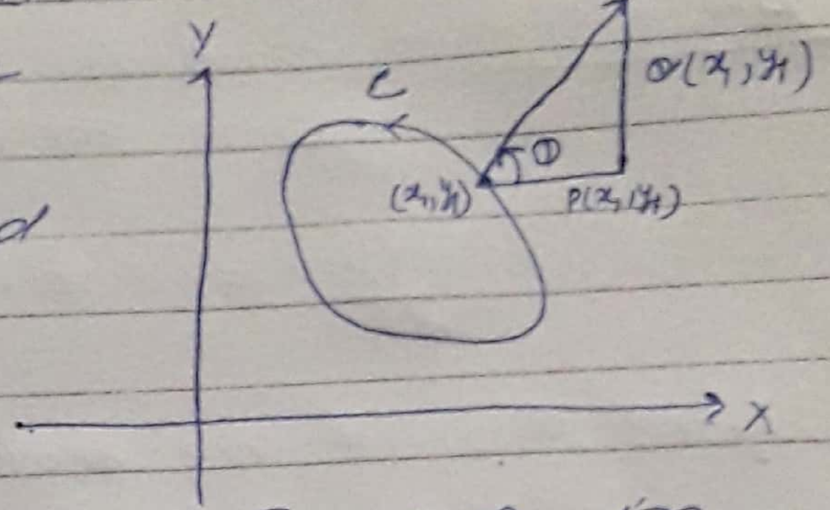
$\therefore$ $C_0$ is a periodic orbit and

$C_0 \subset L(C^+)$

$\Rightarrow$ $L(C^+)$ is a periodic orbit.

**Def:- The Index of a critical Point:-**

Let $\Theta$ denote the angle from the positive x-direction to the vector $[P(x_1,y_1), Q(x_1,y_1)]$ defined by system ① at $(x_1,y_1)$.




Let $\Delta\Theta$ denotes the total change in $\Theta$ as $(x_1,y_1)$ describes the simple closed curve C once in the counter clockwise direction. We call the number $I = \frac{\Delta\Theta}{2\pi}$ the index of the curve C with respect to system ①.

Clearly; $\Delta\Theta$ is either equal to zero or +ive or negative integral multiple of $2\pi$ and hence I is either zero or a +ive integer or a Negative integer. The index for different curve and paths is different. The index of a node, a centre or a spiral point is +1, while index of a saddle point is −1.

Hence the index of a closed path is always +1.

* **Relationship b/w Closed path and Periodic Solution:-**

Consider the autonomous system

$$\frac{dx}{dt} = P(x,y) \quad , \quad \frac{dy}{dt} = Q(x,y) \Big\} \ ①$$

If $x = \phi_1(t)$, $y = \phi_2(t)$ be the periodic solution of equation ① where $\phi_1$ and $\phi_2$ are Non-constant function of t, then $\exists\ T > 0$ st

$$\left.\begin{aligned} \phi_1(t+T) &= \phi_1(t) \\ \phi_2(t+T) &= \phi_2(t) \end{aligned}\right] \rightarrow \text{Solutions are Periodic}$$

Here T = Period of path.

$\Rightarrow \phi_1, \phi_2$ denotes a closed path.

④

$\Rightarrow$ Path is a closed path given by system ①

Conversely: Let $x=\phi_1(t)$ and $y=\phi_2(t)$ be the solution of the system ① satisfying the initial condition $\phi_1(t_0)=x_0$, $\phi_2(t_0)=y_0$.

If we suppose that path is closed, then $\exists$ a value $T>0$ s.t. $t_1=t_0+T$

So that $\phi_1(t_1)=x_0$, $\phi_2(t_1)=y_0$

$\Rightarrow \phi_1(t_0+T)=x_0$, $\phi_2(t_0+T)=y_0$

$\because$ Path is closed
Initially when $t=t_0$
$\phi_1(t_0)=x_0$, $\phi_2(t_0)=y_0$
$\exists$ a $T>0$ s.t.
$t_1=t_0+T$
$\phi_1(t_1)=x_0$, $\phi_2(t_1)=y_0$

Now, consider the systems

$$x=\phi_1(t+T)$$
$$y=\phi_2(t+T)$$

$$\Rightarrow \left.\begin{aligned} \frac{dx}{dt}&=\phi_1'(t+T)\\ \frac{dy}{dt}&=\phi_2'(t+T)\end{aligned}\right\} \quad -\text{⊛}$$

Here $T$ is const.

$\Rightarrow (\phi_1(t+T), \phi_2(t+T))$ is also a solution of the system ①, so that at $t=t_0$, this passes through $(x_0, y_0)$. So, we have two solution of the system ①

But by uniqueness thm., these solutions must be equal

$\therefore \phi_1(t+T)=\phi_1(t)$

and $\phi_2(t+T)=\phi_2(t)$.

$\Rightarrow \phi(t)=(\phi_1(t), \phi_2(t))$ is a periodic function. Thus there is a one-one correspondance b/w closed path and periodic solution.

(5)

**Attractor:-** consider the non-linear system

$$\left.\begin{aligned}\frac{dx}{dt} &= ax + by + f_1(x,y)\\ \frac{dy}{dt} &= cx + dy + f_2(x,y)\end{aligned}\right\} \quad (4)$$

where $a, b, c, d$ are real const. and $f_1, f_2$ are real valued functions defined in the Nbd. of the origin.
If there exist a $\delta > 0$, $0 < \delta < \lambda_0$ s.t. any solution of path $\phi(t) = (\phi_1(t), \phi_2(t))$ of the non-linear system which has atleast point $0 < \xi < \delta$, the solution exists over the t-half line and if solution
$(\phi_1(t), \phi_2(t)) \to (0,0)$ as $t \to \infty$ or $t \to -\infty$.
Then the origin $(0,0)$ is called Attractor of Non-Linear.

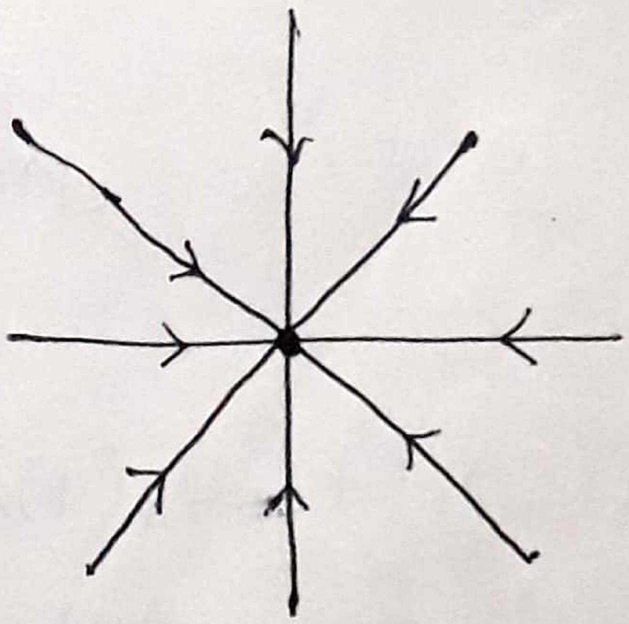
★ Defⁿ of Node in Non-linear systems:-.

The origin is said to be a node if it is attractor for which all orbits arrive at origin in a definite direction. Here attractor shows the path enter (0,0).

★ Proper Node:-.

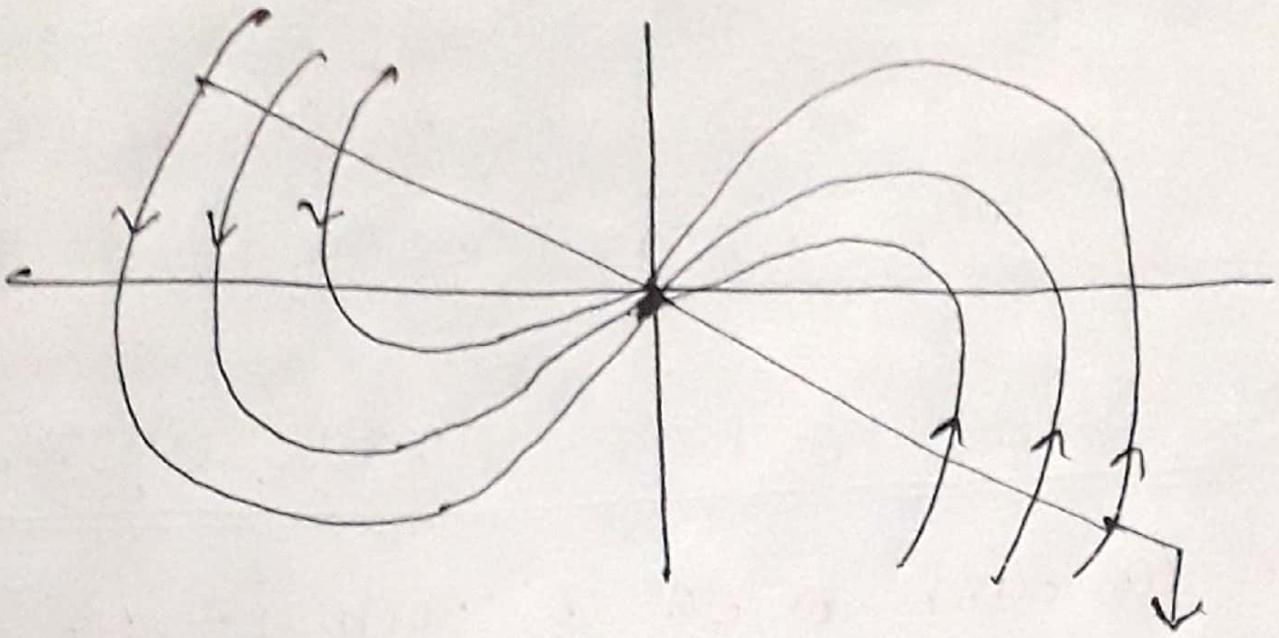
A node is said to be proper if every half line through origin is tangent to the some orbit there.

ex:- A example can be shown with the help of diagram.



★. Improper Node:-.

The node which is not proper i.e; if there exist atleast one half line which is not a tangent to any orbit, then the node is said to be improper node. and improper node can be shown with the help of diagram given below.



This is not a tangent to any orbit.

## *. Spiral point in case of non-linear System.

The origin (0,0) is called spiral pt. of non-linear system

$$\left.\begin{aligned} x' &= ax+by+f_1(x,y) \\ y' &= cx+dy+f_2(x,y) \end{aligned}\right] \longrightarrow (1).$$

If it is attractor s.t. $|w(t)| \to +\infty$ at $t \to \infty$ or $-\infty$.

where $w(t) = \tan^{-1}\left(\dfrac{\phi_2(t)}{\phi_1(t)}\right) \longrightarrow (A).$

where $(\phi_1(t), \phi_2(t))$ is the soln path of the non-linear system which enter the region. $0 \le r < \delta$.