31 March 2020

Diff. Equations. ①
Unit – 3
M.Sc (P) Maths.

Example 2) Determine the stability of the critical point or null solution of

$$\frac{dx}{dt} = -x + y - x^3 - xy^2$$

$$\frac{dy}{dt} = -x - y - y^3 - x^2 y$$

by Liapunov's direct Method.

Sol:- The given Autonomus system

$$\left.\begin{aligned} \frac{dx}{dt} &= -x+y-x^3-xy^2 \\ \frac{dy}{dt} &= -x-y-y^3-x^2y \end{aligned}\right] \quad -①$$

① can be written as

$$\left.\begin{aligned} \frac{dx}{dt} &= -x+y-x[x^2+y^2] \\ \frac{dy}{dt} &= -x-y-y[y^2+x^2] \end{aligned}\right\}$$

Let the function E is defined by

$$E(x,y) = (x^2+y^2) + (y^2+x^2)$$

$$E(x,y) = 2x^2 + 2y^2 \quad -②$$

{ always define E as $E = Ax^2 + By^2$ where A & B are const }

For given system ①, we have

$P = -x+y-x^3-xy^2$, $Q = -x-y-y^3-x^2y$

and for function E, we have

$$\left.\frac{\partial E}{\partial x} = 4x, \quad \frac{\partial E}{\partial y} = 4y\right\} \quad -③$$

Now, we will show that function defined by ② is a Liapunov's function.

Proceeds as in Example ① [last example of pdf 1]

Limit cycles and Periodic Solutions:

(1) Limit cycle :- Consider an Autonomous system

$$\left.\begin{aligned} \frac{dx}{dt} &= P(x,y) \\ \frac{dy}{dt} &= Q(x,y) \end{aligned}\right\} \quad (1)$$

A closed path C of the system (1) which is approached spirally from the inside or the outside by a nonclosed path C_1 of (1) either as $t \to +\infty$ or as $t \to -\infty$ is called a limit cycle of (1)

Example:- Discuss the limit cycle of the system

$$\left.\begin{aligned} \frac{dx}{dt} &= y + x(1-x^2-y^2) \\ \frac{dy}{dt} &= -x + y(1-x^2-y^2) \end{aligned}\right\} \quad (*)$$

Solution:- Using polar Co-ordinates

$x = r\cos\theta$, $y = r\sin\theta$

$$\therefore \frac{dx}{dt} = r(-\sin\theta)\cdot\frac{d\theta}{dt} + (1\cdot\cos\theta)\frac{dr}{dt} \quad (1)$$

$$\frac{dy}{dt} = r(\cos\theta)\cdot\frac{d\theta}{dt} + (\sin\theta)\frac{dr}{dt} \quad (2)$$

Mult. (1) by x and (2) by y, and adding

$$x\frac{dx}{dt} + y\frac{dy}{dt} = x\left[-r\sin\theta\frac{d\theta}{dt} + \cos\theta\frac{dr}{dt}\right] + y\left[r\cos\theta\frac{d\theta}{dt} + \sin\theta\cdot\frac{dr}{dt}\right]$$

$$= (r\cos\theta)\left[-r\sin\theta\frac{d\theta}{dt} + \cos\theta\frac{dr}{dt}\right] + (r\sin\theta)\left[r\cos\theta\cdot\frac{d\theta}{dt} + \sin\theta\cdot\frac{dr}{dt}\right]$$

③

$$x\frac{dx}{dt} + y\frac{dy}{dt} = -r^2\sin\theta\cancel{\cos\theta}\frac{d\theta}{dt} + r\cos^2\theta\frac{dr}{dt}$$
$$+ r^2\sin\theta\cancel{\cos\theta}\frac{d\theta}{dt} + r\sin^2\theta\frac{dr}{dt}$$
$$= r\frac{dr}{dt}(\sin^2\theta + \cos^2\theta) = r\frac{dr}{dt}$$

$$\boxed{x\frac{dx}{dt} + y\frac{dy}{dt} = r\frac{dr}{dt}} \quad -③$$

Now, mult. ① by 'y' and ② by 'x' we get

$$y\frac{dx}{dt} = -r^2\sin^2\theta\frac{d\theta}{dt} + r\sin\theta\cos\theta\frac{dr}{dt}$$

$$x\frac{dy}{dt} = +r^2\cos^2\theta\frac{d\theta}{dt} + r\cos\theta\sin\theta\frac{dr}{dt}$$

$$\Rightarrow x\frac{dy}{dt} - y\frac{dx}{dt} = r^2(\cos^2\theta + \sin^2\theta)\frac{d\theta}{dt}$$

$$\Rightarrow \boxed{x\frac{dy}{dt} - y\frac{dx}{dt} = r^2\frac{d\theta}{dt}} \quad -④$$

Now, from ⊛

$$x\frac{dx}{dt} + y\frac{dy}{dt} = x[y + x(1-x^2-y^2)] + y[-x + y(1-x^2-y^2)]$$
$$= \cancel{xy} + x^2(1-x^2-y^2) - \cancel{xy} + y^2(1-x^2-y^2)$$
$$= (x^2+y^2)(1-x^2-y^2)$$

$$\boxed{x\frac{dx}{dt} + y\frac{dy}{dt} = r^2(1-r^2)} \quad ⑤$$

$\because x = r\cos\theta$
$y = r\sin\theta$
$x^2+y^2 = r^2$

Also, from ⊛

$$x\frac{dy}{dt} - y\frac{dx}{dt} = x[-x + y(1-x^2-y^2)]$$
$$- y[y + x(1-x^2-y^2)]$$
$$= -x^2 + \cancel{xy(1-x^2-y^2)} - y^2 - \cancel{xy(1-x^2-y^2)}$$
$$= -(x^2+y^2)$$

$$\Rightarrow \boxed{x\frac{dy}{dt} - y\frac{dx}{dt} = -r^2} \quad -⑥$$

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Now, from 3 & 5, we have

$$r\frac{dr}{dt} = r^2(1-r^2)$$

$$\Rightarrow \boxed{\frac{dr}{dt} = r(1-r^2)} \quad —(7)$$

and from 4 & 6, we have

$$r^2\frac{d\theta}{dt} = -r^2 \quad \Rightarrow \boxed{\frac{d\theta}{dt} = -1} \quad —(8)$$

Thus the given system in polar form is

$$\frac{dr}{dt} = r(1-r^2), \quad \frac{d\theta}{dt} = -1$$

From (8) $\Rightarrow \frac{d\theta}{dt} = -1 \Rightarrow \int d\theta = \int -dt$

$$\Rightarrow \boxed{\theta = -t + t_0} \rightarrow (9), \; t_0 \text{ is arb. const}$$

From (7) $\Rightarrow \int \frac{dr}{r(1-r^2)} = \int dt \Rightarrow \int \frac{dr}{r(1-r)(1+r)} = \int dt$

$$\Rightarrow \int \left[\frac{1}{r} + \frac{1}{2(1-r)} - \frac{1}{2(1+r)}\right] dr = \int dt$$

[Using Partial fractions.]

$$\Rightarrow \frac{1}{2}\int \left[\frac{2}{r} + \frac{1}{(1-r)} - \frac{1}{(1+r)}\right] dr = \int dt$$

$$\Rightarrow \int \left[\frac{2}{r} + \frac{1}{1-r} - \frac{1}{1+r}\right] dr = \int 2\,dt$$

$$\Rightarrow 2\log r - \log(1-r) - \log(1+r) = 2t + \log C_0$$

$$\Rightarrow 2\log r - [\log(1-r) + \log(1+r)] = 2t + \log C_0$$

$$\Rightarrow \log \frac{r^2}{(1-r)(1+r)} = 2t + \log C_0$$

$$\Rightarrow \frac{r^2}{(1-r^2)} = C_0 e^{2t}$$

$$\Rightarrow r^2 = C_0 e^{2t}[1-r^2]$$

$$\Rightarrow r^2 = C_0 e^{2t} - r^2 C_0 e^{2t}$$

$$\Rightarrow r^2[1 + C_0 e^{2t}] = C_0 e^{2t}$$

$$\Rightarrow \boxed{r^2 = \frac{C_0 e^{2t}}{1 + C_0 e^{2t}}}$$

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change $t_0=0$ then $\theta = -t$.

Sol ⑤

$$r^2 = \frac{1}{\frac{1}{c_0}e^{-2t}+1} = \frac{1}{ce^{-2t}+1}$$ where $\frac{1}{c_0} = c$

$$r = \frac{1}{\sqrt{1+ce^{-2t}}} \quad (10)$$

From ⑨ & ⑩ we get

$r = \frac{1}{\sqrt{1+ce^{-2t}}}$, $\theta = -t + t_0$

where c and t_0 are arb. const.

We may choose $t_0 = 0$, then $\theta = -t$

Solution of system (*) becomes

$$\left.\begin{aligned} x &= r\cos\theta = \frac{1}{\sqrt{1+ce^{-2t}}} \cdot \cos t \\ y &= r\sin\theta = \frac{-1}{\sqrt{1+ce^{-2t}}} \sin t \end{aligned}\right] \quad (**)$$

Define a path of system (*). Let us examine these paths for various value of C.

① If $c=0$; then path defined by $x = \cos t$, $y = -\sin t$ is the circle i.e $x^2+y^2=1$.

② If $c \neq 0$ then the path defined by (**) are not closed path but rather path having a spiral behaviour. [due to exp. funct with c]

ⓐ If $c>0$:- The path are spiral lying inside the circle $x^2+y^2=1$, then $t \to \infty$ approaches the critical point $(0,0)$ of (*)

ⓑ If $c<0$: The path lies outside the circle $x^2+y^2=1$. These outer paths also approaches the circle as $t \to +\infty$.

Thus, the closed path $x^2+y^2=1$ is approached spirally from both the inside and the outside by non-closed paths as $t \to \infty$, we concludes that this circle is a limit cycle.

⑥

Imp Thm Bendixson's NON-existence criteria:-

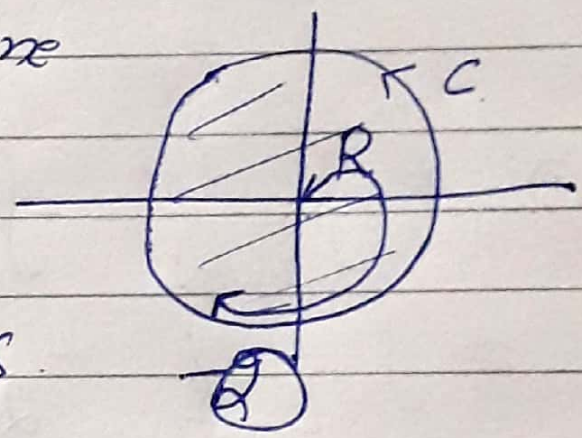
Statement:- Let D be a domain in the xy-plane
Consider the autonomous system

$$\frac{dx}{dt} = P(x,y) \quad , \frac{dy}{dt} = Q(x,y) \quad \Big] \text{ (1)}$$

Where P and Q have its first order partial derivatives in D. Suppose that $\frac{\partial}{\partial x}P(x,y) + \frac{\partial}{\partial y}Q(x,y)$

has the same sign throughout in D. then the System (1) has no closed path in the domain D.

Proof:- Let C be a closed curve in D. Let R be the region bounded by C
∴ By Green's thm. in the Plane we have



$$\int_C [P(x,y)\,dy - Q(x,y)\,dx]$$
$$= \iint_R \left[\frac{\partial P(x,y)}{\partial x} + \frac{\partial Q(x,y)}{\partial y}\right] ds. \quad \text{(2)}$$

Where the line integral is taken in the positive sence.
Let C be the closed path of (1).
Let $x=f(t)$, $y=g(t)$ be an arbitrary sol. of (1) defined by path C parametrically.
Let T denotes the period of this solution.
Then.

$$\left.\begin{aligned} \frac{d f(t)}{dt} &= P[f(t), g(t)] \\ \frac{d g(t)}{dt} &= Q[f(t), g(t)] \end{aligned}\right\} (*)$$

$\Big\{$ By (1)
Put $x=f(t)$
$y=g(t)$

along C and we have

$$\int_C [P(x,y)\,dy - Q(x,y)\,dx] = \int_0^T \left\{ P[f(t),g(t)]\frac{dg(t)}{dt} - Q[f(t),g(t)]\cdot\frac{df(t)}{dt}\right\} dt$$

[on putting $x=f(t)$
$y=g(t)$]

$$\int_C P[x,y]dy - Q(x,y)dx = \int_0^T \{ P[f(t),g(t)]\, Q[f(t),g(t)] - Q[f(t),g(t)]\, P[f(t),g(t)] \} dt$$

$= 0$ [using (*)]

Thus we have, by (2)

$$\iint_R \left[\frac{\partial P(x,y)}{\partial x} + \frac{\partial Q(x,y)}{\partial y} \right] ds = 0$$

But this double integral is zero only if $\frac{\partial P(x,y)}{\partial x} + \frac{\partial Q(x,y)}{\partial y}$ changes sign. This is contradiction to the hypothesis given in the statement.

Thus our supposition is wrong i.e C is a closed path for (1) and hence (1) posses no closed path ~~ma~~.

(*) <u>Half Path</u>:- Consider the autonomous system

$$\frac{dx}{dt} = P(x,y), \quad \frac{dy}{dt} = Q(x,y) \quad \} \quad (1)$$

let C be the path of system (1) and let $x = f(t)$, $y = g(t)$ be a solution of (1) determined by C.

Then the set of all points of C for $t \geq t_0$, where t_0 is same value of t is called a half path of system (1) & is denoted by C^+.

In other words, by a half path of (1) we means the set of all points with coordinates $[f(t), g(t)]$ for $t_0 \leq t < +\infty$.