① [30 March 2020] | Diff. Equation. Section – 3 | CLASS – M.Sc (P) (Maths)

⊛ Liapunov's Direct Method :-

Consider the Non-linear autonomous system $\frac{dx}{dt} = P(x,y)$, $\frac{dy}{dt} = Q(x,y)$

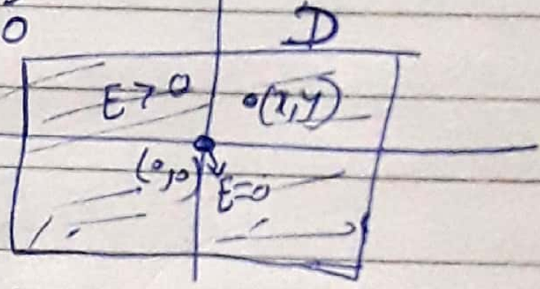
Assume that this system has an isolated critical point at $(0,0)$. [i.e $(0,0)$ is isolated critical point, it is Not a limit point] and P and Q have continous Ist partial derivatives for all (x,y).

Liapunov's method is used for studying the Stability of more general autonomous System.

Def:- let $E(x,y)$ have continous first partial derivatives at all points (x,y) in a domain D containing the point $(0,0)$. Then

(i) The Function E is called +ive definite in D if $E(0,0)=0$ and $E(x,y)>0$ for all other points (x,y) in D. i.e. at $(0,0) \Rightarrow E = 0$

at $(x,y) \neq (0,0) \Rightarrow E > 0$



(ii) The function E is called +ive Semi definite (P.S.D.) in D if $E(0,0)=0$ and $E(x,y) \geq 0$ for all other points (x,y) in D.

i.e at $(0,0)$ $E = 0$

at $(x,y) \neq (0,0)$ $E \geq 0$.

(III) The function E is called Negative definite in D if $E(0,0)=0$ and $E(x,y) < 0$ for all other point in D.

(IV) The function E is called Negative semi definite in D if $E(0,0)=0$ And $E(x,y) \leq 0$ $\forall$ other $(x,y) \in D$.

For e.g. (i) $E(x,y) = x^2+y^2$; $(x,y) \in R^2$

Here, $E=0$ iff $x=0, y=0$

$\therefore$ at $(0,0)$ $E=0$

at any other point $(x,y) \neq (0,0)$ $\Rightarrow E>0$

$\therefore$ E is Positive definite.

(II) $E(x,y) = x^2$; $(x,y) \in R^2$

Here at $(0,0) \Rightarrow$ $E=0$

but if we take $(x,y) = (0,y) \neq (0,0)$

Then $E=0$

Also; if we take $x \neq 0, y \neq 0$

Then $E>0$

$\therefore$ $E(x,y) \geq 0$ for any other point $(x,y) \neq (0,0)$

$\therefore$ E is P.S.D.

Def:- Let $E(x,y)$ have continuous Ist partial derivative at all points (x,y) in a domain D containing $(0,0)$. The derivative of E with respect to system

$$\frac{dx}{dt} = P(x,y) \quad , \frac{dy}{dt} = Q(x,y)$$

is the function $\dot{E}$ (denotes derivt of E) is defined as

$$\boxed{\dot{E}(x,y) = \frac{\partial E(x,y)}{\partial x} \cdot P(x,y) + \frac{\partial E(x,y)}{\partial y} \cdot Q(x,y)}$$

* Liapunov's Function:- Consider the system

$$\frac{dx}{dt} = P(x,y), \quad \frac{dy}{dt} = Q(x,y) \qquad (I)$$

Assume that this system has isolated critical point at origin $(0,0)$ and that P and Q have its first partial derivative for all (x,y)

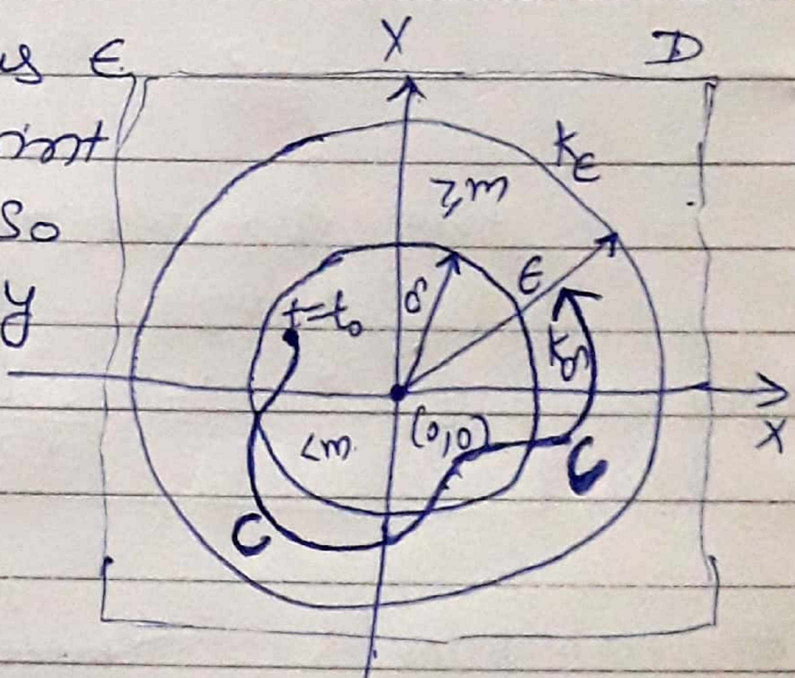
Let $E(x,y)$ be positive definite for all (x,y) in a domain D containing the origin and such that the derivative $\dot{E}(x,y)$ of E with respect to the system ① is Negative semi-definite for all $(x,y) \in D$.
Then E is called Liapunov's Function for system ① in D.

→ Now; we prove two important thms on the stability of critical point $(0,0)$.

V.IMP
Thm 1) Consider the system $\frac{dx}{dt} = P(x,y)$, $\frac{dy}{dt} = Q(x,y)$ ①

Assumes that this system has an isolated critical point at origin $(0,0)$ and P and Q have its first order partial derivative for all (x,y). If there exist a Liapunov's function E for the system ① in some domain D containing $(0,0)$, then the critical point $(0,0)$ of ① is Stable.

Proof:- Let K_ϵ be a circle of radius ϵ with the centre at critical point $(0,0)$. where $\epsilon > 0$ is small enough, so that this circle K_ϵ lies entirely in the domain D



From Real-analysis thm.
" A real-valued function which is continous on a closed bounded set assumes boths a maximum value and minimum value on that set". Since the circle K_ϵ is closed bounded set in the plane and E is continous in D and hence on K_ϵ.

$\therefore$ By Real analysis thm; E must have maximum as well as minimum on K_ϵ.
So; in particular E assumes minimum value on K_ϵ. (ie $E>0$ on K_ϵ) [$\because$ $E>0$ on D]
Thus E assumes a +ive minimum (m) on the circle K_ϵ.
Next: observe that since E is continous at $(0,0)$ and $E(0,0)=0$, $\exists$ a +ive number δ satisfying $\delta<\epsilon$ such that $E(x,y)<m$ for all (x,y) within on circle K_δ of radius δ and centre at $(0,0)$.
Now, let C be any path of (1); let $x=f(t)$, $y=g(t)$ be an arbitrary solution of (1) defined by path 'C' parametrically. and suppose C defined by $[f(t),g(t)]$ is at a point within the "inner" circle K_δ at $t=t_0$. Then

$$E[f(t),g(t)]\Big|_{t=t_0} < m.$$

$$\Rightarrow E[f(t_0),g(t_0)] < m.$$

Since $\dot{E}$ is negative semi-definite in D

$\therefore \frac{d}{dt}E[f(t),g(t)] \le 0$ for $[f(t),g(t)] \in D$.

$\left[\begin{array}{l} \because E \text{ is Liapunov's Function} \\ \Rightarrow E>0 \rightarrow P.D. \\ \text{and } \dot{E} \le 0 \rightarrow N.S.D \end{array}\right]$

Thus $E[f(t),g(t)]$ is non-increasing function of t on C.
[$\because$ $\dot{E}\le 0 \rightarrow$ NON-increasing]

Hence $E[f(t),g(t)] \le E[f(t_0),g(t_0)] < m$ for all $t\ge t_0$

$$\Rightarrow E[f(t),g(t)] < m \quad \forall\ t \ge t_0$$

$\boxed{\text{E is dec. Then for } t\ge t_0\ \ E(t)\le E(t_0)}$

Since $E[f(t),g(t)]$ must have to be $\ge m$ on the outer circle K_ϵ $\therefore$ we see that the path 'C' defined by $x=f(t)$, $y=g(t)$ must remain within K_ϵ for all $t>t_0$. Thus from def. of stability $(0,0)$ is a stable point.

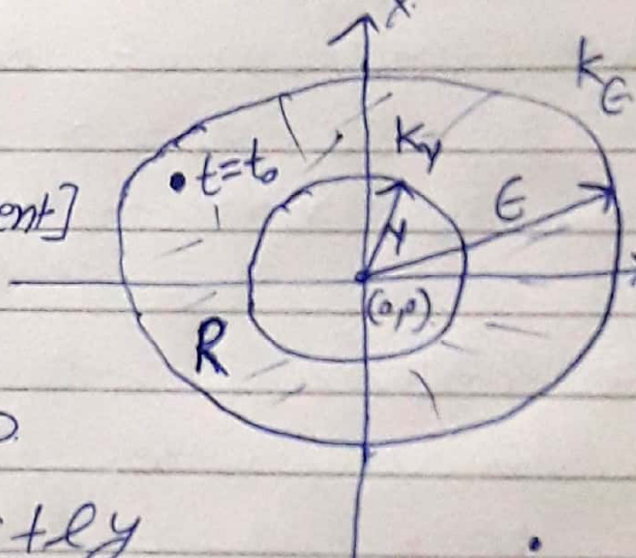
⑤

v.IMP

Thm 2) Consider the system $\frac{dx}{dt} = P(x,y)$, $\frac{dy}{dt} = Q(x,y)$ — (I)

Assume that this system has an isolated critical point at the origin (0,0) and that P and Q have its first partial derivative for all (x,y). If ∃ a Liapunov's function E for the system (I) in some domain D containing (0,0) such that E also has the property that $\dot{E}$ is negative definite in D. then the critical point (0,0) of (I) is asymptotically stable.

Proof:- As in the Prev. thm:, Let K_ϵ be a circle of radius $\epsilon>0$ with centre at the critical point (0,0) and lying entirely in D. Also, let C be any path of (I)
let $x=f(t)$, $y=g(t)$ be any arbitrary sol. of (I) defining C parametrically and suppose 'C' defined by $[f(t), g(t)]$ is at a point within K_ϵ at $t=t_0$



Now, since $\dot{E}$ is negative definite in D [given in statement]

$\therefore \frac{d}{dt} E[f(t), g(t)] < 0$

for $[f(t), g(t)] \in D$

Thus $E[f(t), g(t)]$ is strictly decreasing function of t along C [$\because \dot{E}<0 \Rightarrow E$ is strt. ↓]
Since E is +ive definite [$\because$ E is Liapunov's Funct] in D. $\therefore E[f(t), g(t)] > 0$. for $[f(t), g(t)] \in D$

Thus $\lim_{t\to\infty} E[f(t), g(t)]$ exist and is some no. $L \geq 0$.

i.e $\lim_{t\to\infty} E[f(t), g(t)] = L$

we will show that $L=0$

suppose, if possible let us assume that $L>0$

Since E is +ive definite. There exist a +ive no. γ satisfying $\gamma < \epsilon$ such that $E(x,y) < L$ for all (x,y) within the circle K_γ of radius γ.

Now, applying the result of Real-analysis thm. on maximum and minimum values that we used in the prev. thm. to the continous function $\dot{E}$ on the closed region R between and on the two circles K_ϵ and K_γ.

Since $\dot{E}$ is negative definite in D and hence in this region R which does not includes $(0,0)$.

We see that $\dot{E}$ assumes a negative maximum '$-k$' on R. Since $E[f(t), g(t)]$ is strictley decreasing function of 't' along C and

$$\lim_{t\to\infty} E[f(t), g(t)] = L.$$

The path C defined by $x = f(t)$, $y = g(t)$ cannot enter the domain within K_γ for $t > t_0$ and so remain in R for all $t \geq t_0$

Thus we have $\dot{E}[f(t), g(t)] \leq -k$ for all $t \geq t_0$

Then $\dot{E}$ is negative definite

and we have $\dfrac{d E[f(t), g(t)]}{dt} = \dot{E}[f(t), g(t)] \leq -k$ — (2)

for all $t \geq t_0$

Now consider the Identity

$$E[f(t), g(t)] - E[f(t_0), g(t_0)] = \int_{t_0}^{t} \frac{d}{dt} E[f(t), g(t)]\, dt \quad \text{— (3)}$$

Then (2) gives. [using (2) in (3)]

$$E[f(t), g(t)] - E[f(t_0), g(t_0)] \leq \int_{t_0}^{t} -k\, dt.$$

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$\Rightarrow E[f(t), g(t)] \le E[f(t_0), g(t_0)] - k(t - t_0)$
for all $t \ge t_0$

Now, let $t \to \infty$, since $-k < 0$, we have
$$\lim_{t \to \infty} E[f(t), g(t)] = -\infty$$

$\{\because E[f(t_0), g(t_0)] \downarrow$ fixed point.
$-k < 0$
$t \to \infty$
$-k(t - t_0) = -\infty$

But this contradict the hypothesis that E is +ive definite in D and the assumption that
$$\lim_{t \to \infty} E[f(t), g(t)] = L > 0$$

$\therefore$ our supposition is wrong i.e. $\lim_{t\to\infty} [f(t), g(t)] > 0$
$\therefore L = 0$
$$\Rightarrow \lim_{t \to \infty} E[f(t), g(t)] = 0$$

Since E is positive definite in D, $E(x, y) = 0$ iff $(x, y) = (0, 0)$ [By def. of Positive definite]

Thus
$$\lim_{t \to \infty} E[f(t), g(t)] = 0 \quad \text{if and only if}$$
$$\lim_{t \to \infty} f(t) = 0 \quad \text{and} \quad \lim_{t \to \infty} g(t) = 0$$

But, from the def. of asymptotically stable point; the critical point $(0,0)$ of ① is asymptotical stable.

Example:- Determine the stability of the critical point of
$$\left.\begin{aligned} \frac{dx}{dt} &= -x + y^2 \\ \frac{dy}{dt} &= -y + x^2 \end{aligned}\right\} \quad ①$$

by Liapunov's direct Method.

Solution :- The given autonomous system is

$$\left.\begin{aligned}\frac{dx}{dt} &= -x + y^2 \\ \frac{dy}{dt} &= -y + x^2\end{aligned}\right] \quad (1)$$

let the function E is defined by

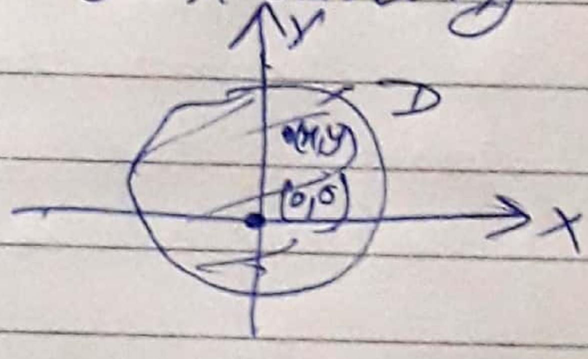
$$E = x^2 + y^2 \qquad -(2)$$

For given system (1) we have $P = -x + y^2$
and $Q = -y + x^2$

and also for function E defined by (2), we have.

$$\left.\frac{\partial E}{\partial x} = 2x \;, \quad \frac{\partial E}{\partial y} = 2y \right\} \quad (3)$$

Now, we will show that function E defined by (2) is a Liapunov's Function.
For given system (1); origin (0,0) is an isolated critical point and P and Q has its first order partial derivates for all $(x,y) \in D$. where D is the Domain for system (1) which containing origin (0,0)



Now; to show E is Liapunov's Function, we have to Prove that.
(i) $E(x,y)$ is a positive definite for all (x,y)
(ii) $\dot{E}(x,y)$ is Negative semi-definite for all $(x,y) \in D$.

Now; (i) $E(x,y) = x^2 + y^2 \quad \forall \; (x,y) \in D$

Here $E = 0$ iff $x = 0, \; y = 0$

$\therefore$ at (0,0) we have $E = 0$

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Also; at other points $(x,y) \neq (0,0) \in D$
we have E is always positive
i.e $E > 0$ for $(x,y) \neq (0,0) \in D$.
$\therefore$ we have
$E = 0$ when $(x,y) = (0,0) \in D$
and $E > 0$ when $(x,y) \neq (0,0) \in D$
Thus E is positive definite for all $(x,y) \in D$.

(ii)
We know that $\dot{E}(x,y) = \frac{\partial E}{\partial x} \cdot P(x,y) + \frac{\partial E}{\partial y} \cdot Q(x,y)$

$= 2x \cdot [-x+y^2] + 2y[-y+x^2]$

$= -2x^2 + 2xy^2 - 2y^2 + 2yx^2$

$\dot{E}(x,y) = -2(x^2+y^2) + 2(x^2y + xy^2)$ — (4)
for all $(x,y) \in D$

Now; if $\dot{E}$ is negative semi-definite for all (x,y) in some domain D containing $(0,0)$, then E defined by (2) is a Liapunove function for system (1)
Clearly; $\dot{E}(0,0) = 0$
Now observe the following; if $x<1$ and $y \neq 0$, then $xy^2 < y^2$; if $y<1$ and $x \neq 0$, then $x^2y < x^2$.
Thus if $x<1, y<1$ and $(x,y) \neq (0,0)$
Then $x^2y + yx^2 < x^2+y^2$
$\Rightarrow 2(x^2y + yx^2) < 2(x^2+y^2)$
$\Rightarrow 2(x^2y+yx^2) - 2(x^2+y^2) < 0$.
$\Rightarrow \dot{E}(x,y) < 0$ [Using (4)]

Thus in every domain D containing $(0,0)$ and such that $x<1$ and $y<1$, $\dot{E}(x,y)$ given by (4) is Negative semi definite.
Hence, E defined by (2) is Liapunove function
$\Rightarrow$ By thm. (1) $(0,0)$ is a stable point.