FERMI-DIRAC GAS DEGENERACY:

fermi gas is an assembly of fermions obeying F-D statistics i.e

$$n_i = \frac{g_i}{e^{\alpha} e^{\beta u_i} + 1}$$ (derived in last lecture)

No. of particles in momentum range p and p+dp

$$n(p)\,dp = \frac{g(p)\,dp}{A^{-1} e^{u/KT} + 1}$$

as $e^{\alpha} = A^{-1}$ and $\cancel{\beta = \frac{1}{E}}$ $\beta = \frac{1}{KT}$

[and $g(p)\,dp = g_s \frac{4\pi V p^2\,dp}{h^3}$]

Please Note:

$g_s = 2s+1$, which is spin degeneracy factor due to spin s of fermions.

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Hence

$$n(p)\,dp = g_s \frac{4\pi V p^2}{h^3} \frac{1}{A^{-1} e^{u/KT}+1}\,dp$$

As $u = \frac{p^2}{2m}$ $\Rightarrow$ $2p\,dp = 2m\,du$
(differentiating both sides)

$$\Rightarrow dp = \frac{m\,du}{p} = \frac{m\,du}{\sqrt{2mu}} = \left(\frac{m}{2u}\right)^{1/2} du$$

$$\therefore n(u)\,du = \frac{g_s \frac{4\pi V}{h^3} (2mu) \left(\frac{m}{2u}\right)^{1/2} du}{A^{-1} e^{u/KT}+1}$$

(we have replaced p by u here)

$$n(u)\,du = g_s \frac{4\pi V}{h^3} \frac{(2u)^{1/2} m^{3/2}}{A^{-1} e^{u/KT}+1}\,du$$

$x = \frac{u}{KT}$ $\qquad$ $u = KTx$

$du = KT\,dx$

$$\Rightarrow n(u)\,du = g_s \frac{4\pi V}{h^3} (m)^{3/2} \frac{2^{1/2} (KTx)^{1/2} KT\,dx}{A^{-1} e^{x}+1}$$

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$$n(u)du = g_s V \left(\frac{2\pi m K T}{h^2}\right)^{3/2} \frac{2}{\sqrt{\pi}} \frac{x^{1/2}dx}{A^{-1}e^{x}+1}.$$

Now total number of particles is given by

$$n = \int_0^\infty n(u)du$$

$$= g_s V \left(\frac{2\pi m K T}{h^2}\right)^{3/2} \frac{2}{\sqrt{\pi}} \int_0^\infty \frac{x^{1/2}dx}{A^{-1}e^{x}+1} \quad \text{— (1)}$$

And Hence

Total energy of system is given by

$$E = \int_0^\infty n(u)du \times u$$

$$= KT g_s V \left(\frac{2\pi m K T}{h^2}\right)^{3/2} \frac{2}{\sqrt{\pi}} \int_0^\infty \frac{x \cdot x^{1/2}dx}{A^{-1}e^{x}+1}$$

$$E = g_s \frac{3V}{2} \left(\frac{2\pi m K T}{h^2}\right)^{3/2} KT \frac{4}{3\sqrt{\pi}} \int_0^\infty \frac{x^{3/2}dx}{A^{-1}e^{x}+1}$$

These 2 equations are solved under 2 conditions:

1.) **Weak degeneracy:**

When α is positive and $A<1$, this condition is called "weak degeneracy".

So from equ — (1)

$$\frac{2}{\sqrt{\pi}}\int_0^\infty \frac{x^{1/2}\,dx}{A^{-1}e^{x}+1}$$

$$= \frac{2}{\sqrt{\pi}}\int_0^\infty \frac{x^{1/2}\,dx}{A^{-1}e^{x}(1+Ae^{-x})}$$

(Taking $A^{-1}e^{x}$ common)

$$= \frac{2}{\sqrt{\pi}}\int_0^\infty Ae^{-x}\,x^{1/2}\,dx\,(1+Ae^{-x})^{-1}$$

(Rearranging)

$$= \frac{2}{\sqrt{\pi}}\left[A\int_0^\infty x^{1/2}e^{-x}\left[1-Ae^{-x}+A^2e^{-2x}+\cdots\right)dx\right.$$

(expanding $(1+x)^{-1} = 1-x+x^2-x^3+\cdots$)

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$$= \frac{2}{\sqrt{\pi}}\left[A\int_0^\infty x^{1/2}e^{-x}dx - A^2\int_0^\infty x^{1/2}e^{-2x}dx + \cdots\right]$$

$$= \frac{2}{\sqrt{\pi}}\left[\frac{\sqrt{\pi}}{2}A - \frac{\sqrt{\pi}}{2}\frac{A^2}{2^{3/2}} + \frac{A^3}{3^{3/2}} - \cdots\right]$$

$$\left(\text{Using } \int_0^\infty x^n e^{-ax}dx = \frac{n!}{a^{n+1}}\right)$$

$\therefore$ we get

$$\frac{2}{\sqrt{\pi}}\int_0^\infty \frac{x^{1/2}dx}{A^{-1}e^x+1} = \left(A - \frac{A^2}{2^{3/2}} + \frac{A^3}{3^{3/2}} - \cdots\right)$$

Similarly we can evaluate

$$\frac{4}{3\sqrt{\pi}}\int_0^\infty \frac{x^{3/2}dx}{A^{-1}e^x+1} = \left(A - \frac{A^2}{2^{5/2}} + \frac{A^3}{3^{5/2}} - \cdots\right)$$

Hence in weak degeneracy we get

$$n = g_s V\left(\frac{2\pi mkT}{h^2}\right)^{3/2}\left(A - \frac{A^2}{2^{3/2}} + \frac{A^3}{3^{3/2}} - \cdots\right)$$

$$E = g_s \frac{3V}{2}\left(\frac{2\pi mkT}{h^2}\right)^{3/2} kT\left(A - \frac{A^2}{2^{5/2}} + \frac{A^3}{3^{5/2}} - \cdots\right)$$

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dividing these 2 equs:

$$\frac{E}{n} = \frac{3}{2}KT \frac{\left(A - \frac{A^2}{2^{5/2}} + \frac{A^3}{3^{5/2}} - \cdots\right)}{\left(A - \frac{A^2}{2^{3/2}} + \frac{A^3}{3^{3/2}} - \cdots\right)}$$

$$= \frac{3}{2}KTA\left(1 - \frac{A}{2^{5/2}} + \frac{A^2}{3^{5/2}} - \cdots\right) \Big/ \left(A - \frac{A^2}{2^{3/2}} + \frac{A^3}{3^{3/2}} - \cdots\right)$$

(Rearranging)

Now taking A^{-1} common from right term

we get

$$\frac{E}{n} = \frac{3}{2}KT\left(1 - \frac{A}{2^{5/2}} + \frac{A^2}{3^{5/2}} - \cdots\right)\left(1 + \frac{A}{2^{3/2}} - \frac{A^2}{3^{3/2}} - \cdots\right)$$

$$\frac{E}{n} = \frac{3}{2}KT\left[1 + \frac{A}{2^{5/2}} - \frac{A^3}{3^{5/2}} + \cdots\right]$$

For $A \ll 1$ Higher powers are neglected

Thus $\boxed{\frac{E}{n} = \frac{3}{2}KT}$

degeneracy is weak Here.

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2) **Strong Degeneracy:**

when α is negative and $A>>1$, the degeneracy will become prominant..

so from -①

equ. 1 becomes.

$$n = g_s V \left(\frac{2\pi mKT}{h^2}\right)^{3/2} A$$

[all other terms neglected]

$$\Rightarrow A = \frac{1}{g_s}\left(\frac{h^2}{2\pi mKT}\right)^{3/2} \frac{n}{V}.$$

so the gas $\longrightarrow$ highly degenerate if $\frac{n}{V}$ is large and temp $\longrightarrow$ low.