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Arbitrary Series

* The infinite series in which terms are Connected by both positive and negative sign

Dirichlet's Test

Statement If $\sum_{n=1}^{\infty} a_n$ is a series of real numbers whose sequence $\langle S_n \rangle$ of partial sums is bounded and if $\langle b_n \rangle$ is a non negative monotonically decreasing sequence leading to zero then the series $\sum_{n=1}^{\infty} a_n b_n$ converges

Proof Let $S_n = a_1 + a_2 + \cdots + a_n$

As sequence $\langle S_n \rangle$ of partial sum is bounded therefore there exist a +ve integer k such that

$$|S_n| \le k \quad \forall\, n \in N$$

$$S_n = a_1 + a_2 + \cdots + a_n$$

$$S_{n+p} = a_1 + a_2 + \cdots + a_n + a_{n+1} + \cdots + a_{n+p}$$

$$S_{n+p} - S_n = a_{n+1} + a_{n+2} + \cdots + a_{n+p}$$

$$\begin{aligned} |a_{n+1} + a_{n+2} + \cdots + a_{n+p}| &= |S_{n+p} - S_n| \\ &\le |S_{n+p}| + |S_n| \quad [\because |a-b| \le |a| + |b|] \\ &\le k + k \\ &= 2k \end{aligned}$$

$\langle S_n \rangle$ is given to be bounded, $n \in N$

so $|S_n| \le k$ and $|S_{n+p}| \le k$

Also b_n converges to zero
By def. of convergence

For given $\varepsilon > 0$, there exist a +ve integer m such that

$$|b_n - 0| < \frac{\varepsilon}{2k} \quad \forall\ n \geq m$$

$$b_n < \frac{\varepsilon}{2k} \quad \forall\ n \geq m$$

Applying Abel's lemma, we get

$$|a_{n+1} b_{n+1} + a_{n+2} b_{n+2} + \cdots + a_{n+p} b_{n+p}| < 2k\, b_{n+1}$$
$$< 2k \frac{\varepsilon}{2k}$$
$$= \varepsilon$$

[Statement of Abel's lemma
if $\sum_{n=1}^{\infty} a_n$ is a series of real numbers such that $m \leq a_1 + a_2 + \cdots + a_n \leq M$ and $\langle b_n \rangle$ is a non decreasing sequence of non negative real numbers such that $m b_1 \leq \sum_{k=1}^{n} a_k b_k \leq M b_1 \ \forall\ n \in N$]

By Cauchy general principle of Convergence $\sum_{n=1}^{\infty} a_n b_n$ convergent

[Statement of Cauchy General Principle of Convergence
A necessary and sufficient condition for convergence of series $\sum_{n=1}^{\infty} a_n$ is that to $\varepsilon > 0$ there exist a +ve integer m such that
$|a_{m+1} + a_{m+2} + \cdots + a_n| < \varepsilon$ whenever $n > m$]

~~For / understanding /~~

For understanding
Here $|a_{n+1} + a_{n+2} + \dots + a_{n+p}| \le 2k$ and in Abel's lemma here the term m and M are multiplied by least element and $\forall\, n \geq m$ $b_n < \frac{\epsilon}{2k}$
b_{n+1} is least so $2k$ is multiplied by b_{n+1}

so $b_{n+1} < \frac{\epsilon}{2k}$

example

Test the convergence of the series $\sum_{n=1}^{\infty} \frac{(-1)^n \sin n\alpha}{n^p}$, $p > 0$

for all real α

Solution The given series $\sum_{n=1}^{\infty} \frac{(-1)^n \sin n\alpha}{n^p}$, $p > 0$

$$= \sum_{n=1}^{\infty} a_n b_n \quad (\text{say}$$

$$a_n = (-1)^n \sin n\alpha \qquad \text{and } b_n = \frac{1}{n^p}$$

[we know that
$\sin(n\pi + \theta) = (-1)^n \sin\theta$]

Now $a_n = \sin(n\pi + n\alpha) = (-1)^n \sin n\alpha$

$$S_n = a_1 + a_2 + \dots + a_n$$
$$= \sin(\pi + \alpha) + \sin(2\pi + 2\alpha) + \dots + \sin(n\pi + n\alpha)$$
$$= \frac{\sin\left(\frac{\pi + \alpha + n\pi + n\alpha}{2}\right) \sin\left(n\left(\frac{\pi + \alpha}{2}\right)\right)}{\sin\left(\frac{\pi+\alpha}{2}\right)}$$

Case I

[$\because \sin\theta_1 + \sin\theta_2 + \dots + \sin\theta_n = \dfrac{\sin\left(\frac{\text{First angle} + \text{Last angle}}{2}\right) \sin\left(\frac{n \cdot \text{diff of angle}}{2}\right)}{\sin\left(\frac{\text{difference of angle}}{2}\right)}$]

For understanding
[we use here Dirichlet's Test because it is easy to show partial sum which involve sine and cosine term to bounded $|\sin\theta| \le 1$ and $|\cos\theta| \le 1$]

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$$|S_n| = \left| \frac{\sin\left(\frac{\pi+\alpha+n\pi+n\alpha}{2}\right) \sin\left(\frac{n(\pi+\alpha)}{2}\right)}{\sin\left(\frac{\pi+\alpha}{2}\right)} \right|$$

$$= \frac{\left|\sin\left(\frac{\pi+\alpha+n\pi+n\alpha}{2}\right)\right| \left|\sin\left(\frac{n(\pi+\alpha)}{2}\right)\right|}{\left|\sin\left(\frac{\pi}{2}+\frac{\alpha}{2}\right)\right|}$$

$$\leq \frac{1 \cdot 1}{\left|\cos\frac{\alpha}{2}\right|} \qquad \left[\because \sin\left(\frac{\pi}{2}+\theta\right) = \cos\theta\right] \quad \text{——(2)}$$

Here two case arises

Case I

$$\left|\cos\frac{\alpha}{2}\right| \neq 0$$

$$\left|\cos\frac{\alpha}{2}\right| \neq 0$$

$$\frac{\alpha}{2} \neq (2k+1)\frac{\pi}{2} \quad \text{where } k \text{ is any integer}$$

$n\left(\frac{n \cdot \text{diff of angle}}{2}\right)$

$$\alpha \neq (2k+1)\pi$$

From (2)

$$|S_n| \leq \text{Finite quantity}$$

$\Rightarrow$ Sequence $\langle S_n \rangle$ of partial sum is bounded

and Here $b_n = \frac{1}{n^p}$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n^p} = \frac{1}{\infty} = 0$$

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Also $(n+1) > n \quad \forall n$

$(n+1)^p > n^p$ for $p > 0$

$\frac{1}{(n+1)^p} < \frac{1}{n^p} \quad \forall n, p > 0$

$b_{n+1} < b_n \quad \forall n$

$\Rightarrow$ Sequence $\langle b_n \rangle$ is decreasing and converges to zero — (3)

Then by Dirichlet's Test $\sum_{n=1}^{\infty} a_n b_n$ is convergent

Case II – when $\alpha = (2k+1)\pi$ where k is any integer

$\sin n\alpha = 0$

The given series $\sum_{n=1}^{\infty} \frac{(-1)^n \sin n\alpha}{n^p} = \sum_{n=1}^{\infty} \frac{(-1)^n \cdot 0}{n^p} = 0$

which is convergent.