B.A./B.Sc. - 2nd Semester

Ordinary Differential Equations

Chapter - 6

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METHOD - II

(Solution of 2nd Order Linear Differential Equations by removing first derivative and changing the dependent variable.)

In this method it is clear from its name that we have to perform two tasks:-
→ Change the dependent variable i.e. y in eqn. (1)
&
→ Remove the First derivative (dy/dx) term.

Conside 2nd order L.D.E.:-

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \longrightarrow (1)$$

~~Chang~~ Let $y = u.v \rightarrow (2)$

We have to solve the D.E. (1) i.e. we have to find value of y.

We have supposed $y = u.v$ that is now we have to find u & v, put them in equation (2), and we will get y as solution.

In method-I discussed in previous lecture we have studied different ways of finding u, but in this method we choose u such that by taking this value the first derivative term get removed.

~~And for this~~ And from article 6.4 of your book you can check that when we take $u = e^{-\frac{1}{2}\int P dx}$ then first derivative terms get removed automatically.

And we obtain a new D.E. in v.

Working Rules :-

Step 1 :- Compare given eq^n. with equation

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

and obtain values of P, Q, R.

You can also use method-I here but if you see that value of u is difficult to find from P, Q, R then you can use this method.

Step 2 :- put $y = uv$ in the given equation.

Then Find

(i) $u = e^{-\frac{1}{2}\int P dx}$,

$$P' = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx}$$

and

$$Q' = \frac{R}{u}.$$

By using, ~~the~~ $u = e^{-\frac{1}{2}\int P\,dx}$, given eq^n. transforms into another form $\frac{d^2v}{dx^2} + P'v = Q'$

and ~~so~~ when we solve this equation, we will get value of v.

Now multiplying u & v gives required solution i.e. $\underline{\underline{y = uv}}$. Ans.

Note (i) When it is difficult for you to find value of u that P, Q, R satisfies which equation

Note (ii) When you see a statement of the question, then you can apply Method-I as well as method-II.

But you have to chose which one should be used.

In step-I, you find values of P, Q & R. Check wheather it is easy for you to find u by using the relations

→ $1+P+Q=0 \Rightarrow u=e^{x}$

→ $1-P+Q=0 \Rightarrow u=e^{-x}$

→ $1+\frac{P}{m}+\frac{Q}{m^2}=0 \Rightarrow u=e^{mx}$.

→ $P+Qx=0 \Rightarrow u=x$

→ $2+2Px+Qx^2=0 \Rightarrow u=x^2$

→ $m(m-1)+mPx+Qx^2=0 \Rightarrow u=x^m$. etc.

But if above relations are not satisfied by P, Q, R or it is difficult to find then you can use this method.

(ii) We apply Method-II when P' is either constant or of the form $\frac{\text{constant}}{x^2}$.

∴ These values make the equations easy to solve further.

if P' = const. then you get an eq^n. with constant coefficients & can be solved by Chapter-4.

& if $P'=\frac{\text{const.}}{x^2}$, then resulting equation will be homogeneous & will be solved by Chapter-5.

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Example) Given $\frac{d^2y}{dx^2} - 2\tan x \frac{dy}{dx} + 5y = 0$

Solve by removing first ~~Method~~ derivative

Solution, $\frac{d^2y}{dx^2} - 2\tan x \frac{dy}{dx} + 5y = 0 \rightarrow (1)$

Compare (1) with

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

$\Rightarrow P = -2\tan x$, $Q = 5$, $R = 0$

It is clear by viewing P, Q, R that value of u is not easy to find by using method-1.

So we used Method-II. Also it is even mentioned in the statement that solve by using removing first derivative.

put $y = u \cdot v$

where $u = e^{-\frac{1}{2}\int P dx} = e^{-\frac{1}{2}\int -2\tan x\, dx}$

$= e^{\int \tan x} = e^{\log \sec x} = \sec x$.

Notes

$$P' = Q - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{4} = 5 - \frac{1}{2}\frac{d}{dx}(-2\tan x) - \frac{1}{4}(4\tan^2 x)$$

$= 5 + \sec^2 x - \tan^2 x = 5 + (1 + \tan^2 x) - \tan^2 x$

$= 6$ = constant.

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$$Q' = \frac{R}{u} = \frac{0}{u} = 0$$

Now when we take $u = e^{-\frac{1}{2}\int P\,dx}$ & $y = u \cdot v$

then equation (1) reduces to new eq^n :-

$$\frac{d^2 v}{dx^2} + P' v = Q' \longrightarrow (2)$$

put values of P' & Q' :-

$$\frac{d^2 v}{dx^2} + 6v = 0 \longrightarrow (3)$$

which is a L.D.E. with constant coefficients & we can solve it by using methods of chapter (4) :-

(3) $\Rightarrow (D^2+6)v = 0$

Auxiliary eq^n. is $D^2+6=0 \Rightarrow D = \pm\sqrt{6}\,i$

$\Rightarrow v = C_1 \cos\sqrt{6}\,x + C_2 \sin\sqrt{6}\,x$

Hence required solution is $y = u \cdot v$

i.e. $y = \sec x\,(C_1 \cos\sqrt{6}\,x + C_2 \sin\sqrt{6}\,x)$ Ans.

Notes

Note $\Rightarrow$ We can note one thing here that $P' =$ constant gives eq^n. (3) which is L.D.E. with constant coefficients (Chapter-4)

But if $P' = \frac{\text{const.}}{x^2}$ say $\frac{6}{x^2}$ then eq^n. (3) becomes

$$\frac{d^2 v}{dx^2} + \frac{6}{x^2}v = 0 \Rightarrow x^2\frac{d^2 v}{dx^2} + 6v = 0 \rightarrow$$ This is homo. form i.e. can be solved by chapter-5.