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B.A./B.Sc. - 2nd Sem.

Ordinary Differential Equations

Chapter - 6

"Linear Differential Equations of Second Order."

As it is clear from the name of the chapter that the equations will be
→ Linear Differential Equations (L.D.E.)
(As studied in chapter No - 4 & 5)
→ Order will always be 2.
→ Degree will be one ($\because$ Linear D.E.)

In chapter - 4, we were dealing with L.D.E. in which their coefficients are constant and we studied different methods to solve them.

In chapter - 5, we dealt with L.D.E. which are homogeneous and now we know how to solve them.

In this chapter, we will deal with those ~~(second order)~~ L.D.E. whose coefficients are neither constant nor homogeneous. And in addition to this the order of D.E. is also fixed ~~and~~ it is 2.

Introduction :-

The general form of linear Diff. equation of second order is :-

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

→ Order is 2 here
→ Degree is one
→ Multiplication of dependen variable i.e. y with itself and with its derivatives is not present.
} Linear D.E.

→ Further here P, Q and R are functions of x only.

If P, Q are constants, then methods of chapter-4 will apply here.

But if P, Q are not constants & not homogeneous then we can have these five methods :-

(1) By changing the dependent variable when an integral included in the C.F. is known.
(2) By removing the Ist derivative and changing the dependent variable.
(3) By changing the independent variable.
(4) By method of variation of Parameters
(5) By method of undetermined coefficients.

M	T	W	T	F	S	S
			1	2	3	4
5	6	7	8	9	10	11
12	13	14	15	16	17	18
19	20	21	22	23	24	25
26	27	28	29	30	31	

Method-I → By changing the dependent variable when an integral included in the C.F. is known :- ↓

Let the second order L.D.E. is:-

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \longrightarrow (1)$$

Working Rule:-

Step 1:- Make the coefficient of $\frac{d^2y}{dx^2}$ one.

Step 2 → In this method we change the dep. variable i.e. y by putting

~~and~~ $y = u.v \rightarrow (*)$ (y is replaced by product of u & v.)

Sometimes u is given in the statement and sometimes we have to find it.

After Working Rules, I will mention the rules to find u.

Step 3 → Since y is changed into u & v so from eq^n. (1) also, we will eliminate y as follows:-

$y = uv$

we will find $\frac{dy}{dx}$ & $\frac{d^2y}{dx^2}$ and put in equation (1).

After putting these values we finally get an eq^n. of the form

$$\frac{d^2 v}{dx^2} + P' \frac{dv}{dx} = Q' \longrightarrow (2)$$

Step 4 $\rightarrow$ In eq^n. (2), when we solve it we will get the value of v

and put this value of v in eq^n. (*) i.e. $y = u \cdot v$ which is our required solution.

~~In tt~~

But from eq^n. (2), we will find v by using a substitution,

we put $\frac{dv}{dx} = p$, $\frac{d^2 v}{dx^2} = \frac{dp}{dx}$

$$(2) \Rightarrow \frac{dp}{dx} + P' p = Q' \longrightarrow (3)$$

Now we finally get an eq^n. of Ist order & we have many ways to solve it studied earlier.

Sol^n. of Eq^n. (3) gives value of p.

by using $p = \frac{dv}{dx}$, we will get v.

& using $y = u \cdot v$, we will get y i.e. solution of given D.E. (1).

M	T	W	T	F	S	S
			1	2	3	4
5	6	7	8	9	10	11
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Rules to find u (u represents integral involved in the C.F.)

$\boxed{A}$ $y = e^{mx}$ is a sol^n of ① if $1 + \frac{P}{m} + \frac{Q}{m^2} = 0$

i.e. if values of P & Q given in eq^n ①

satisfies $1 + \frac{P}{m} + \frac{Q}{m^2} = 0$

then we take $u = e^{mx}$.

(i) put $m = 1$ in $y = e^{mx}$ i.e. $y = e^x$ is a sol^n if $1 + \frac{P}{1} + \frac{Q}{1^2} = 0$ i.e. $1 + P + Q = 0$

Means if $1 + P + Q = 0$ then we take $u = e^x$

(ii) for $m = -1$ i.e. $1 - P + Q = 0 \Rightarrow u = e^{-x}$

$\boxed{B}$ $y = x^m$ is a sol^n of ① then

$$m(m-1) + P.mx + Qx^2 = 0$$

(i) put $m=1 \Rightarrow 1(1-1) + P.1.x + Qx^2 = 0 \Rightarrow Px + Qx^2 = 0$

$\Rightarrow P + Qx = 0$

i.e. if $P + Qx = 0$ then we take $u = x^1 = x$.

(ii) $m = 2 \Rightarrow 2(2-1) + 2Px + Qx^2 = 0$

$\Rightarrow 2 + 2Px + Qx^2 = 0$

i.e. if $2 + 2Px + Qx^2 = 0$ then $u = x^m = x^2$

Similarly, we can put $m = 3, 4, ----$ and so on.

Questions Based on this Method

Q1 Solve the D.E. :-

$$x^2 \frac{d^2y}{dx^2} - (x^2+2x)\frac{dy}{dx} + (x+2)y = x^3 e^x \longrightarrow (*)$$

Solution We have to solve this equations i.e. we have to find value of y.

Firstly, In eq^n (*) make coefficient of $\frac{d^2y}{dx^2}$ one by dividing throughout by x^2 :-

$$\frac{x^2}{x^2}\frac{d^2y}{dx^2} - \left(\frac{x^2+2x}{x^2}\right)\frac{dy}{dx} + \frac{(x+2)}{x^2}y = \frac{x^3 \cdot e^x}{x^2}$$

$$\Rightarrow \frac{d^2y}{dx^2} - \left(1+\frac{2}{x}\right)\frac{dy}{dx} + \left(\frac{1}{x}+\frac{2}{x^2}\right)y = xe^x \longrightarrow (1)$$

Now put $y = u.v \longrightarrow (**)$

Our next purpose is just to find u & v. After ~~get~~ finding u & v we put these values in (**) & get the answer.

First we find u :-

Compare eq^n (1) with $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$

we see that

$$P = -1 - \frac{2}{x}, \quad Q = \frac{1}{x} + \frac{2}{x^2}, \quad R = xe^x.$$

Now by viewing P, Q we can easily see that

$$P + Qx = -1 - \frac{2}{x} + \frac{x}{x} + \frac{2}{x} = 0$$

i.e. if I multiply Q with x & add in P then equation $P + Qx$ is satisfied. Then from Rule B (part i), we have

$$y = x \text{ is a part of C.F.}$$

i.e. $\boxed{u = x}$

Now we have find the value of u, remaining process will be used to find value of v.

Since $y = u \cdot v$

$\Rightarrow y = xv \rightarrow (3^*)$ Step-3 of Working Rules.

$$\therefore \frac{dy}{dx} = \frac{dv}{dx}x + v$$

$$\& \frac{d^2y}{dx^2} = \frac{d^2v}{dx^2}x + \frac{dv}{dx} + \frac{dv}{dx} = \frac{d^2v}{dx^2}x + 2\frac{dv}{dx}$$

Notes

Now put value of $y, \frac{dy}{dx}$ & $\frac{d^2y}{dx^2}$ in equation (1) :-

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$$\Rightarrow \frac{d^2u}{dx^2}x + 2\frac{du}{dx} - \left(1+\frac{2}{x}\right)\left(\frac{du}{dx}x + u\right) + \left(\frac{1}{x}+\frac{2}{x^2}\right)u \cdot x = xe^x$$

$$\Rightarrow \frac{d^2u}{dx^2}x + \cancel{2\frac{du}{dx}} - \frac{du}{dx}x - \cancel{2\frac{du}{dx}} - \cancel{u} - \frac{2u}{x} + \cancel{\frac{1}{x}} + \cancel{u} + \frac{2u}{x} = xe^x.$$

$$\Rightarrow x\frac{d^2u}{dx^2} - x\frac{du}{dx} = xe^x$$

$$\Rightarrow \frac{d^2u}{dx^2} - \frac{du}{dx} = e^x \longrightarrow (1*)$$

You can see here, by using $y = u.v$ we have changed dep. variable y in terms of u & v and equation (1) got the new form as written above in (1*).

In (1*) dep. var. is now u.

Now we find u from this equation.

For finding u, put $\frac{du}{dx} = p$ & $\frac{d^2u}{dx^2} = \frac{dp}{dx}$

$$(1*) \Rightarrow \frac{dp}{dx} - p = e^x \longrightarrow (2)$$

which is a linear differential equation of form $\frac{dp}{dx} + P'p = Q'$ where $P' = -1$
$Q' = e^x$.

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As you know,
~~of form So~~ for solving (2), we find

$$I.F. = e^{\int P' dx} = e^{\int -1 dx} = e^{-x}$$

& solution of (2) will be:-

$$p(I.F.) = \int Q(I.F.)\, dx + \text{Constant}$$

$$\Rightarrow p e^{-x} = \int e^{x} \cdot e^{-x} dx + C_1$$

or $$p e^{-x} = \int dx + C_1$$

or $$p e^{-x} = x + C_1$$

or $$p = x e^{x} + C_1 e^{x}$$

Value of p is obtained but we need v:-

So as we had ~~ats~~ put $p = \frac{dv}{dx}$

$$\Rightarrow \frac{dv}{dx} = p \Rightarrow \frac{dv}{dx} = x e^{x} + C_1 e^{x}$$

To find v, integrate it:-

$$v = \int \underset{I}{x} \underset{II}{e^{x}} dx + \int C_1 e^{x}$$

$$v = x e^{x} - e^{x} + C_1 e^{x} + C_2$$

Now we have taken $y = u \cdot v$

$$\Rightarrow y = x v \quad \text{(from eq}^n\text{. 3*)}$$

$$\Rightarrow y = x(x e^{x} - e^{x} + C_1 e^{x} + C_2)$$ Ans.