B.A./B.Sc. - 4th Semester
Subject Mathematics
(Prog. in C and Numerical Methods)
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"Applications of Cholesky Method"

Introduction:- Cholesky method is used to solve simultaneous linear algebraic equations as we have already discussed. This method is also known as Square Root Method. Further this method is used only for symmetric and positive definite matrices.

We have studied that we decompose matrix A as product of lower triangular matrix L & its transpose L'. (system of eqⁿs. is expressed as AX=B)

i.e. $A = LL'$ $\rightarrow (1)$

and in the previous lecture you have seen that by comparing the corresponding elements of eqⁿ. (1) we can find elements of matrix L i.e.

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$$

& hence L' can also be found ~~by~~ from L i.e.

$$L' = \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

Now we are with ~~known~~ matrix L whose all elements are known to us.

Application:- By using Cholesky or Square Root Method we can find inverse of a matrix. With this value of L we can find inverse of any symmetric & positive definite matrix.

Suppose we have to find ~~A~~ inverse of matrix A i.e. A^{-1}.

and since $A = LL'$

$$\Rightarrow A^{-1} = (LL')^{-1} = (L')^{-1} L^{-1} = (L^{-1})' L^{-1}$$

Thus for finding A^{-1}, we need product of two matrices $(L^{-1})'$ and L^{-1} i.e. we have to basically find L^{-1} & from here its transpose $(L^{-1})'$ can be found easily.

So for finding inverse we have to follow following steps :-

(1) Decompose A into L & L' i.e. $A = LL'$ (A should be symmetric & positive definite)

(2) Find elements of matrix L.

(3) Find inverse of L i.e. L^{-1}

(4) Solve $A^{-1} = (L^{-1})' . L^{-1}$.

Question :- Find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 5 & 10 \\ 4 & 10 & 21 \end{bmatrix}$

by Cholesky / Square Root Method.

Solution Let $A = LL' \longrightarrow (1)$

where $L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$ $\Rightarrow$ $L' = \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$

put values in (1) :-

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 5 & 10 \\ 4 & 10 & 21 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 4 \\ 2 & 5 & 10 \\ 4 & 10 & 21 \end{bmatrix} = \begin{bmatrix} l_{11}^2 & l_{11}.l_{21} & l_{11}.l_{31} \\ l_{21}.l_{11} & l_{21}^2 + l_{22} & l_{21}l_{31} + l_{22}l_{32} \\ l_{31}.l_{11} & l_{31}l_{21} + l_{32}l_{22} & l_{31}^2 + l_{32}^2 + l_{33}^2 \end{bmatrix}$$

Now comparing the corresponding elements :-

① Ist Row :- $l_{11}^2 = 1 \Rightarrow l_{11} = 1$

$$l_{11} \cdot l_{21} = 2 \Rightarrow l_{21} = 2$$

$$l_{11} \cdot l_{31} = 4 \Rightarrow l_{31} = 4$$

② 2nd Row :- $l_{21}^2 + l_{22}^2 = 5 \Rightarrow l_{22} = 1$

$$l_{21} \cdot l_{31} + l_{22} \cdot l_{32} = 10 \Rightarrow l_{32} = 2.$$

③ 3rd Row :- $l_{31}^2 + l_{32}^2 + l_{33}^2 = 21 \Rightarrow l_{33} = 1$

$$\therefore \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix}$$

Now we will find L^{-1}. L^{-1} is some another lower triangular matrix ($\because$ inverse of a Lower Triag. Matrix is also a lower triang. Matrix).

$$\text{Let } L^{-1} = \begin{bmatrix} l_{11}^* & 0 & 0 \\ l_{21}^* & l_{22}^* & 0 \\ l_{31}^* & l_{32}^* & l_{33}^* \end{bmatrix}$$

Since $LL^{-1} = I$ ($\because$ Any Matrix on multiplication with its inverse give Identity Matrix.)

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} l_{11}^* & 0 & 0 \\ l_{21}^* & l_{22}^* & 0 \\ l_{31}^* & l_{32}^* & l_{33}^* \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

On Comparing Corresponding Elements, we get l_{ij}^* as follows:

① Ist Column $l_{11}^* = 1$

$$2 l_{11}^* + l_{21}^* = 0 \Rightarrow l_{21}^* = -2$$

$$4 l_{11}^* + 2 l_{21}^* + l_{31}^* \Rightarrow l_{31}^* = 0$$

② 2nd Column $l_{22}^* = 1$ & $4 \cdot 0 + 2 \cdot l_{22}^* + l_{32}^* = 0 \Rightarrow l_{32}^* = -2.$

③ 3rd Column $1 \cdot l_{33}^* = 1$

$$\Rightarrow l_{33}^* = 1$$

In this way all the elements of L^{-1} are known to us.

i.e. $L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ & hence $(L^{-1})' = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$ (4)

and hence $A^{-1} = (LL')^{-1} = (L')^{-1} L^{-1} = (L^{-1})' L^{-1}$

$$= \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -2 & 0 \\ -2 & 5 & -2 \\ 0 & -2 & 1 \end{bmatrix}$$

Thus Cholesky / Square Root Method also helps us ~~to~~ in finding inverse of a matrix.