Chapter → 3

Legender's Equation :→

The differential equation of the form

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$

is Called Legender's equation, where n is Called the Parameter, which May be real or Complex.

1. Solution of Legender's Equation

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$

First we Compare with general form

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\,y = 0$$

Here we find P(x) and Q(x), then Check the point $x=0$ is O.P and S.P. Then proceed the Method accordingly.

Legendre's Polynomial →

We have seen that the solⁿ of Legendre's equation is Linear Combination of two independent solution $y_1(x)$ and $y_2(x)$ where.

$$y_1(x) = 1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!}x^4 - \quad \text{--- (1)}$$

$$y_2(x) = x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 + \quad \text{--- (2)}$$

In n is integer one of (1) and (2) will terminate.

If n is +ve even integer then series (1) will terminate at the terms x^n because all the succeeding terms vanish. and the series $y_1(x)$ becomes a polynomials. Similarly if n is +ve odd integer, then the series (2) will terminate at x^n and $y_2(x)$ becomes a polynomials.

Similarly in case of -ve integer one of (1) and (2) will becomes a polynomials.

Thus we see that whenever n is an integer, positive or negative, the general solution of Legendre's equation Consist of a polynomials solution (finite terms) and infinite Series solution.

Polynomials solution with a_0 or a_1 Choosen in such a way that the value of the polynomial became 1 at $x=1$ is called Legendre's Polynomial of order n and is denoted by $P_n(x)$.

Non-terminating infinite series solution with a_0 ~~and~~ or a_1 suitably choosen is called Legendre's function of IInd kinds and it denoted by $Q_n(x)$.

So we discuss here two types of Legendre's Polynomials.

2 H.Imp Rodrigue's Formula

OR

To Prove that $P_n(x) = \frac{1}{2^n \cdot n!} \frac{d^n}{dx^n}(x^2-1)^n$

Pf → Let $y = (x^2-1)^n$ — (1)

Diff w.r.t to x, both side we have,

$$y_1 = n(x^2-1)^{n-1} \cdot 2x$$

$$(x^2-1)y_1 = 2xn(x^2-1)^n$$

$$(x^2-1)y_1 = 2nxy \qquad \text{(by using (1))}$$

Differentiating $(n+1)$ times by Lebnitz's theorem.

$$^{n+1}C_0\, y_{n+2}(x^2-1) + {}^{n+1}C_1\, y_{n+1} \cdot 2x + {}^{n+1}C_2\, y_n \cdot 2$$

$$= 2n\left[{}^{n+1}C_0\, y_{n+1} \cdot x + {}^{n+1}C_1\, y_n\right]$$

$$\Rightarrow y_{n+2}(x^2-1) + (n+1)y_{n+1} \cdot 2x + \frac{n(n+1)}{\cancel{2}}\cancel{2}y_n$$

$$= 2n\left[y_{n+1} \cdot x + (n+1)y_n\right]$$

$$(x^2-1)y_{n+2} + xy_{n+1}(\cancel{2n} + 2 - \cancel{2n}) - n(n+1)y_n =$$

$$(1-x^2)y_{n+2} - 2xy_{n+1} + n(n+1)y_n = 0 \quad \text{— (2)}$$

Eq^n (2) suggest that y_n satisfy the Legendre's equation $(1-x^2)y_2 - 2xy_1 + n(n+1)y = 0$

But we know Legendre's eqⁿ has $P_n(x)$ and $Q_n(x)$ as its solution. But $y_n = \frac{d^n}{dx^n}(x^2-1)^n$ contain only +ve power of x.

And if finite series solution is $P_n(x)$, then it can be only some constant multiple of C.

$$P_n(x) = c\,y_n = c\frac{d^n}{dx^n}(x^2-1)^n$$

$$P_n(x) = c\frac{d^n}{dx^n}(x-1)^n(x+1)^n.$$

Differentiating the R.H.S by product rule using Leibnitz theorem, we have.

$$P_n(x) = c\left[{}^nC_0 \frac{d^n}{dx^n}(x-1)^n.(x+1)^n + {}^nC_1 \frac{d^{n-1}}{dx^{n-1}}(x-1)^n.n(x+1)^{n-1} + \cdots (x+1)^n.\, n!\right]$$

In order to find c, we take $x=1$ and $P_n(1)=1$, we have

$$1 = c\left[1.n!.2^n + 0 + 0 - - - + 0\right]$$

[∵ nth derivative of $(x-1)^n$ is $n!$

$$c = \frac{1}{2^n.n!}$$

Thus $P_n(x) = \frac{1}{2^n.n!}\frac{d^n}{dx^n}(x^2-1)^n$

This is required Rodrigue's formula.