B.A./B.Sc. - 2nd Semester
Ordinary Differential Equations
Chapter - 6

(By Kanak Sharma
Deptt. of Mathematics
I.B. College, Panipat.)

# Method - 3.

(Solution of Linear Differential equations of Second Order by changing the Independent Variable.)

Consider the standard form of 2nd order LDE :-

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \longrightarrow (1)$$

where, P, Q, R are functions of $x$ only.

In Method - I & II, we have changed the dependent variable i.e. $y$ but in this method we will change the independent variable i.e. $x$.

Thus there we have changed $y$ into $u$ & $v$ i.e. $y \begin{cases} \to u \\ \to v \end{cases}$

and we formed a new D.E. in $v$ i.e. from eq$^n$ (1) we eliminated $y$. — In Method - I, II.

Now in this method we change $x$ and bring new independent variable $z$ and we will form a new equation in which $x$ is not there and $z$ will act as independent variable.

Working Rules:

Step-I → Bring the given D.E. in the standard form

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \rightarrow (2)$$

Compare and find values of P, Q, R.

Step 2:- Now we have to introduce $z$ in place of $x$, i.e. we have to find a relationship between $x$ & $z$. This relationship can be found by two ways. You can use any one.

$\boxed{A}$ Find $z$ from $\frac{dz}{dx} = e^{-\int P dx}$

and $z = \int \frac{dz}{dx} dx = \int e^{-\int P dx} dx$.

If we use this method of finding $z$, then always $P_1$ (will be discussed later in step-3) comes out to zero i.e. $P_1 = 0$ here.

$\boxed{B}$ If it is difficult to find $z$ from the above point A, then we have another way of finding $z$ by using

$$\left(\frac{dz}{dx}\right)^2 = \frac{|Q|}{a^2} \rightarrow$$ $|Q|$ means positive value of $Q$ is used whether it is +ve or -ve.

where $a^2$ is a constant to be chosen.
~~We choose~~ $\cancel{\frac{dz}{dx}}$ ~~in such a way that $P_1$ comes out to be~~

**Step 3:-** Now by using point A or point B we will have a new variable $z$ and we get a new equation with the help of eq$^n$. ① in which $x$ get removed & $z$ get introduced. And this eq$^n$. is:-

$$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1 \longrightarrow ②$$

where $P_1 = \dfrac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}$, $Q_1 = \dfrac{Q}{\left(\frac{dz}{dx}\right)^2}$

and $R_1 = \dfrac{R}{\left(\frac{dz}{dx}\right)^2}$

Thus we have eq$^n$. ② in $z$ as independent variable.
Solve eq$^n$. ② & find value of $y$ which is in terms of $z$.

Notes

**Step 4:-** Put value of $z$ in terms of $x$ in the solution obtained in step 3.
Thus we have finally value of $y$ in terms of $x$.

**Note** If we use point A for finding $z$, then $P_1$ always will be zero & hence eq$^n$ (2) will take the form:–

$$\frac{d^2y}{dz^2} + Q_1 y = R_1 \qquad \left(\because P_1 = 0, \text{ find } Q_1 \,\&\, R_1\right)$$

and if we use point B for finding $z$, then $P_1$ ~~need~~ can take any value zero or other than zero, in that case eq$^n$ (2) will remain the same and we have to find $P_1$, $Q_1$, $R_1$ all. One thing to be noticed here is that in $\left(\frac{dz}{dx}\right)^2 = \frac{|Q|}{a^2}$, we choose $a^2$ such that $P_1$ comes out to be a constant because constant value of $P_1$ will make eq$^n$ (2) easy to solve.

**Question**:– Solve:–

$$x\frac{d^2y}{dx^2} + (4x^2-1)\frac{dy}{dx} + 4x^3 y = 2x^3$$

Divide throughout by $x$:–

$$\frac{d^2y}{dx^2} + \left(4x - \frac{1}{x}\right)\frac{dy}{dx} + 4x^2 y = 2x^2 \longrightarrow (1)$$

Comparing with $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$, we have

$P = 4x - \frac{1}{x}$ , $Q = 4x^2$ , $R = 2x^2$.

Now using point A, find z :-

$$\frac{dz}{dx} = e^{-\int P dx} = e^{-\int \left(4x - \frac{1}{x}\right) dx} = e^{-(2x^2 - \log x)}$$

$$\Rightarrow \frac{dz}{dx} = e^{-2x^2 + \log x} = e^{-2x^2} \cdot e^{\log x}$$

$$\Rightarrow \frac{dz}{dx} = e^{-2x^2} \cdot x \longrightarrow (*)$$

Integrate to find z :- $z = \int e^{-2x^2} \cdot x \, dx$

$$\Rightarrow z = -\frac{1}{4} \int e^{-2x^2} (-4x) \, dx$$

$$= -\frac{1}{4} e^{-2x^2} \qquad \left[\because \int e^{f(x)} \cdot f'(x) \, dx = e^{f(x)}\right]$$

Thus we get a relationship in z & x i.e. $z = -\frac{1}{4} e^{-2x^2}$. We use this relation to change independent variable x to z in eq^n (1) which is given D.E. & we get a new eq^n :-

$$\frac{d^2y}{dz^2} + P_1 \frac{dy}{dz} + Q_1 y = R_1 \longrightarrow (2)$$

where $P_1 = 0$,

$$Q_1 = \frac{|Q|}{\left(\frac{dz}{dx}\right)^2} = \frac{4x^2}{x^2 \cdot e^{-4x^2}} \qquad \left(\left(\frac{dz}{dx}\right)^2 = x^2 e^{-4x^2} \text{ from } (*)\right)$$

$$Q_1 = \frac{4}{e^{-4x^2}} = \frac{4}{16z^2}$$

Convert from $x$ to $z$.

$\because Z = -\frac{1}{4} e^{-2x^2}$

$\Rightarrow -4z = e^{-2x^2}$

square both sides

$\Rightarrow 16z^2 = e^{-4x^2}$

$$\Rightarrow Q_1 = \frac{1}{4z^2}$$

and $$R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = \frac{2x^2}{x^2 \cdot e^{-4x^2}} = \frac{2}{e^{-4x^2}} = \frac{2}{16z^2}$$

$$\Rightarrow R_1 = \frac{1}{8z^2}$$

Hence eqn. 2 becomes,

$$\frac{d^2y}{dz^2} + \frac{1}{4z^2} y = \frac{1}{8z^2}$$

(Note :- Sometimes this eqn. will be in the form of L.D.E. with constant coefficients, then we can use chapter-4 and sometimes it will be in the form of Homogeneous L.D.E. then we will use chapter-5 and sometimes take some another form which you can solve.)

Multiply above eqn. by $z^2$ :-

$$z^2\frac{d^2y}{dz^2} + \frac{1}{4} y = \frac{1}{8} \longrightarrow (3)$$

which is homogeneous L.D.E. and can be solved by using methods studied in chapter-5.

put $z = e^t$ so that $\log z = t$

$\therefore z\frac{d}{dz} = \frac{d}{dt} = D$ & $z^2\frac{d^2}{dz^2} = D(D-1)$

from (3), $\left(D(D-1)+\frac{1}{4}\right)y = \frac{1}{8}$

or $\left(D^2 - D + \frac{1}{4}\right)y = \frac{1}{8}$

A.E. $D^2 - D + \frac{1}{4} = 0 \Rightarrow D = \frac{1}{2}, \frac{1}{2}$

$\therefore$ C.F. $= (c_1 + c_2 t)e^{\frac{1}{2}t} = (c_1 + c_2 \log z)z^{1/2}$

& P.I. $= \frac{1}{\frac{1}{4} - D + D^2}\left(\frac{1}{8}\right)e^{0t} = \frac{1}{2}$

Hence solution of (3) is

$y = (c_1 + c_2 \log z)z^{1/2} + \frac{1}{2}$

convert it in $x$:-

$y = (c_1 + c_2 x^2)e^{-x^2} + \frac{1}{2}$ Ans.

Alternative Method Using pt. B of working Rules

Choose $z$ s.t. $\left(\frac{dz}{dx}\right)^2 = \frac{Q}{a^2} = 4x^2$ (choose $a^2 = 1$)

$\Rightarrow \frac{dz}{dx} = 2x \Rightarrow z = \int 2x\,dx = x^2.$

Now changing the Indep. variable from $x$ to $z$ by using $z = x^2$, we get

$$\frac{d^2y}{dz^2} + P_1 \frac{dy}{dz} + Q_1 y = R_1 \longrightarrow (2) \quad \left(\text{Here } P_1 \neq 0 \text{ trivially}\right)$$

where $P_1 = \dfrac{\dfrac{d^2z}{dx^2} + P\dfrac{dz}{dx}}{\left(\dfrac{dz}{dx}\right)^2} = \dfrac{2 + \left(4x - \frac{1}{x}\right)(2x)}{4x^2}$

$$= \frac{\cancel{2} + 8x^2 - \cancel{2}}{4x^2} = 2$$

(Note in Previous point A process, $P_1$ was zero and we did not find $P_1$ but here in point B process ~~$P_1$~~ we have to find $P_1$ $\because$ it is not ~~zero~~ necessary that $P_1$ will be 0.)

$$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = \frac{4x^2}{4x^2} = 1$$

$$\& \; R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = \frac{2x^2}{4x^2} = \frac{1}{2}$$

put values in (2) :- $\dfrac{d^2y}{dz^2} + 2\dfrac{dy}{dz} + y = \dfrac{1}{2}$

L.D.E. with constant coefficients.

A.E. $D^2 + 2D + 1 = 0 \Rightarrow D = 1, -1$

C.F. $= (c_1 + c_2 z)\, e^{-z}$



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$$P.I. = \frac{1}{D^2+2D+1} \cdot \frac{1}{2} e^{0z} = \frac{1}{2}.$$

Hence solution is $y = C.F. + P.I.$

$$\Rightarrow \quad y = (c_1 + c_2 z)\, e^{-z} + \frac{1}{2}$$

convert in $x$:-

$$y = (c_1 + c_2 x^2)\, e^{-x^2} + \frac{1}{2} \quad \text{Ans.}$$

Notes