Mobius Transformation (continued)

Thm ⇒ Prove that resultant of two Mobius transformations is again a Mobius transformation.

proof ⇒ Let

$$w = \frac{az+b}{cz+d} \quad \text{such that } ad-bc \neq 0 \quad \rightarrow (1)$$

and,

$$\phi(w) = \frac{a_1 w + b_1}{c_1 w + d_1} \quad \text{such that } a_1 d_1 - b_1 c_1 \neq 0 \quad \rightarrow (2)$$

be the two Mobius transformation.

Claim ⇒ $\phi(w(z))$ is again Mobius transformation.

so, For this put the value of w in (2),

$$\phi(w(z)) = \frac{a_1\left(\frac{az+b}{cz+d}\right)+b_1}{c_1\left(\frac{az+b}{cz+d}\right)+d_1}$$

$$= \frac{aa_1z + a_1b + b_1cz + b_1d}{ac_1z + bc_1 + d_1cz + dd_1}$$

$$= \frac{(aa_1+b_1c)z + (a_1b+b_1d)}{(c_1a+d_1c)z + (dd_1+c_1d)}$$

$$\phi(w(z)) = \frac{Az+B}{Cz+D} \quad (\text{say})$$

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where, $A = a_1 a + b_1 c$
$B = a_1 b + b_1 d$
$C = c_1 a + d_1 c$
$D = d_1 d + c_1 d$

Thus, $\phi(w(z)) = \dfrac{Az+B}{Cz+D}$

It will be a Mobius transformation, if $AD - BC \neq 0$

<u>To Check</u> $\Rightarrow$ $AD-BC$

$= (aa_1 + b_1 c)(dd_1 + c_1 b) - (a_1 b + b_1 d)(c_1 a + d_1 c)$

$= aa_1(dd_1 + c_1 b) + b_1 c(dd_1 + c_1 b)$
$- a_1 b(c_1 a + d_1 c) - b_1 d(c_1 a + d_1 c)$

Taking a_1 common from 1st and 3rd term, (and b_1 common from 2nd and 4th term),

$= a_1[add_1 + \cancel{ac_1 b} - \cancel{bc_1 a} - bd_1 c]$
$+ b_1[c\cancel{dd_1} + c c_1 b - adc_1 - c\cancel{dd_1}]$

$= a_1 d_1(ad - bc) + b_1 c_1(bc - ad)$

$= a_1 d_1(ad - bc) - b_1 c_1(ad - bc)$

$= (a_1 d_1 - b_1 c_1)(ad - bc) \neq 0$

$\left[\because (a_1 d_1 - b_1 c_1) \neq 0 \ \& \ (ad - bc) \neq 0\right)$

$\Rightarrow \boxed{AD - BC \neq 0}$

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Thus, $\phi(w(z))$ is a Mobius Transformation, i.e., resultant of two mobius transformations is again Mobius.

Thm $\Rightarrow$ Prove that every Mobius transformation is the resultant of Mobius transformations.

Proof $\Rightarrow$ Let $w = \frac{az+b}{cz+d}$; $ad-bc \neq 0$ and $c \neq 0$

be any mobius transformation, then

$$w = \frac{a\left(z+\frac{b}{a}\right)}{c\left(z+\frac{d}{c}\right)}$$

$$= \frac{a}{c}\left[1 + \frac{\frac{b}{a}-\frac{d}{c}}{z+\frac{d}{c}}\right]$$

(By solving, $\frac{z+\frac{b}{a}}{z+\frac{d}{c}} = 1 + \frac{\frac{b}{a}-\frac{d}{c}}{z+\frac{d}{c}}$)

$$= \frac{a}{c} + \frac{a}{c}\left[\frac{bc-ad}{ac}\right] \cdot \frac{1}{z+\frac{d}{c}}$$

$$w = \frac{a}{c} + \frac{bc-ad}{c^2} \cdot \frac{1}{\left(z+\frac{d}{c}\right)}.$$

Let $\boxed{z+\frac{d}{c} = z_1}$

then; $w = \frac{a}{c} + \frac{bc-ad}{c^2}\left(\frac{1}{z_1}\right)$

Let $\frac{1}{z_1} = z_2$

then, $w = \frac{a}{c} + \frac{bc-ad}{c^2}(z_2)$

Let $\boxed{z_2\left(\frac{bc-ad}{c^2}\right) = z_3}$

$\Rightarrow$ $\boxed{w = z_3 + \frac{a}{c}}$

translation

Clearly, $w = z + \alpha$ type

or $w = \frac{1}{z}$ inversion type

or, $w = \beta z$ type

(By z_1, z_2, z_3 forms)

Hence, Mobius transformation is the resultant of above types of Mobius transformations.

Cross Ratio :→

Let z_1, z_2, z_3, z_4 be the distinct points, then the ratio

$$\frac{(z_1-z_2)\ (z_3-z_4)}{(z_2-z_3)\ (z_4-z_1)}$$ is called

the cross ratio of z_1, z_2, z_3, z_4 denoted by (z_1, z_2, z_3, z_4).

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Note ⇒ For cross ratio of four points you need only to write differences of points i.e, z_1-z_2, z_2-z_3, z_3-z_4, z_4-z_1 in cyclic order, and put them in num. then in denominator alternatively.

i.e,
$$\frac{(z_2-z_3)(z_4-z_1)}{(z_1-z_2)(z_3-z_4)}$$

Que ⇒ Find cross ratio of $(0, 1, i, -1)$

Sol^n ⇒
$$\frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)}$$

$$= \frac{(0-1)(1+i)}{(1-i)(-1-0)} = \frac{1+i}{1-i} = \frac{1+i}{1-i} \times \frac{1+i}{1+i}$$

$$= \frac{(1+i)^2}{1+1}$$

$$= \frac{\cancel{1}-\cancel{1}+2i}{2} = \boxed{i}$$

Thm ⇒ Show that cross ratio remains invariant under Mobius Transformation.

Proof ⇒ We have to prove, cross ratio remains invariant, i.e, remains the same after the transformation also.

Hence, let $w = \frac{az+b}{cz+d}$; $ad-bc \neq 0$

be any Mobius Transformation, which transforms z_1, z_2, z_3, z_4 to the corresponding pts w_1, w_2, w_3 and w_4 in the w-plane.

Claim :→ $\dfrac{(z_2-z_3)(z_4-z_1)}{(z_1-z_2)(z_3-z_4)} = \dfrac{(w_2-w_3)(w_4-w_1)}{(w_1-w_2)(w_3-w_4)}$

Clearly, $w_r = \dfrac{az_r+b}{cz_r+d}$ $\left(\because w = \dfrac{az+b}{cz+d}\right)$

$\Rightarrow w_r - w_s = \dfrac{az_r+b}{cz_r+d} - \dfrac{az_s+b}{cz_s+d}$

$= \dfrac{(az_r+b)(cz_s+d) - (az_s+b)(cz_r+d)}{(cz_r+d)(cz_s+d)}$

$= \dfrac{az_r(cz_s+d) + b(cz_s+d) - az_s(cz_r+d) - b(cz_r+d)}{(cz_r+d)(cz_s+d)}$

Taking a common from 1st and 3rd term, ||ly b from 2nd & 4th.

$= \dfrac{a[c\cancel{z_r z_s} + z_r d - c\cancel{z_r z_s} - z_s d] + b[cz_s + \cancel{d} - cz_r - \cancel{d}]}{(cz_r+d)(cz_s+d)}$

$= \dfrac{ad(z_r - z_s) + bc(z_s - z_r)}{(cz_r+d)(cz_s+d)}$

$$\Rightarrow w_r - w_s = \frac{(ad-bc)(z_r - z_s)}{(cz_r+d)(cz_s+d)}$$

For $r = 1, 2, 3, 4$ and $s = 1, 2, 3, 4$ above eqn. will become,

$$w_1 - w_2 = \frac{(ad-bc)(z_1 - z_2)}{(cz_1+d)(cz_2+d)}$$

$$w_3 - w_4 = \frac{(ad-bc)(z_3 - z_4)}{(cz_3+d)(cz_4+d)}$$

$$w_3 - w_2 = \frac{(ad-bc)(z_3 - z_2)}{(cz_3+d)(cz_2+d)}$$

$$w_1 - w_4 = \frac{(ad-bc)(z_1 - z_4)}{(cz_1+d)(cz_4+d)}$$

Now, $$(w_1 - w_4)(w_3 - w_2) = \frac{(ad-bc)^2 (z_1 - z_4)(z_3 - z_2)}{(cz_3+d)(cz_2+d)(cz_1+d)(cz_4+d)}$$

Illy, $$(w_1 - w_2)(w_3 - w_4) = \frac{(ad-bc)^2 (z_1 - z_2)(z_3 - z_4)}{(cz_1+d)(cz_2+d)(cz_3+d)(cz_4+d)}$$

dividing above two eqns (to get cross ratio)

$$\frac{(w_1 - w_4)(w_3 - w_2)}{(w_1 - w_2)(w_3 - w_4)} = \frac{(z_1 - z_4)(z_3 - z_2)}{(z_1 - z_2)(z_3 - z_4)}$$

i.e, $(w_1, w_2, w_3, w_4) = (z_1, z_2, z_3, z_4)$

i.e, Cross ratio remains invariant.