① 31 March 2020

Vector Calculus
B.A / B.SC – Ist Year
TOPIC – Line Integral (Part 2)

**Example 4)** Evaluate $\int_C \vec{f} \cdot d\vec{r}$ where $\vec{f} = y\hat{i} + (x+z)^2\hat{j} + (x-z)^2\hat{k}$

from $(0,0,0)$ to $(2,4,0)$ along
(i) the straight line $y = 2x$ in $xy$-plane.
(ii) the parabola $y = x^2$, $z = 0$

**Solution:–** (i) Along the straight line $y = 2x$
$\Rightarrow dy = 2dx$ —— (*)

As integration is performed in $xy$-plane

$\therefore \vec{r} = x\hat{i} + y\hat{j}$ [Here $z = 0$ in $xy$-plane]
$\Rightarrow d\vec{r} = dx\hat{i} + dy\hat{j}$
$d\vec{r} = dx\hat{i} + 2dx\hat{j}$ [using (*)]

Also; on straight line $y = 2x$, so $\vec{f}$ function
$\vec{f} = y\hat{i} + (x+z)^2\hat{j} + (x-z)^2\hat{k}$
becomes
$\vec{f} = 2x\hat{i} + x^2\hat{j} + x^2\hat{k}$ [putting $y = 2x$, $z = 0$]

Now

$$\int_C \vec{f} \cdot d\vec{r} = \int_0^2 [2x\hat{i} + x^2\hat{j} + x^2\hat{k}] \cdot [dx\hat{i} + 2dx\hat{j}]$$

$$= \int_0^2 [2x\,dx + 2x^2\,dx]$$

$$= \int_0^2 (2x + 2x^2)\,dx$$

$$= \left[\frac{2x^2}{2} + \frac{2x^3}{3}\right]_0^2$$

$$= \left[x^2 + \frac{2x^3}{3}\right]_0^2 = \left[4 + \frac{16}{3} - 0\right] = \boxed{\frac{28}{3}}$$

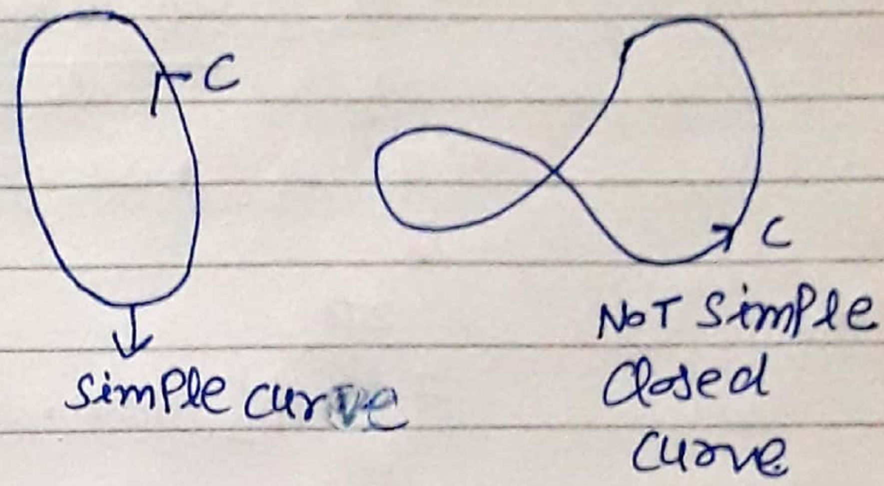
[limit of integration
C → $(0,0,0)$ to $(2,4,0)$
$x$ varies 0 to 2
$y$ varies 0 to 4
$z$ varies 0 to 0]

(ii) Similarly in part (ii) the curve is
Parabola $y = x^2$, $z = 0$
Here $dy = 2x\,dx$

$\therefore \vec{r} = x\hat{i} + y\hat{j} \Rightarrow d\vec{r} = dx\hat{i} + dy\hat{j}$
$= dx\hat{i} + 2x\,dx\hat{j}$

Do similar as in part (i)

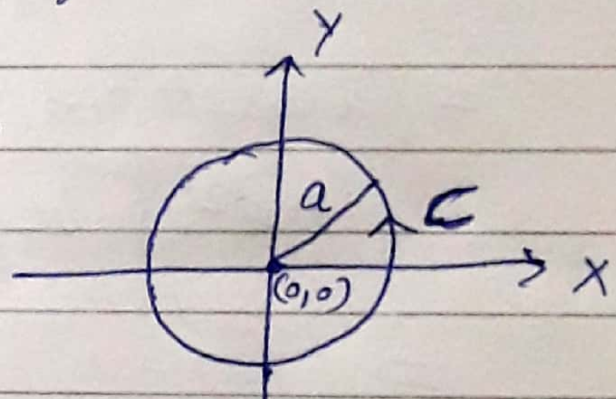
**Def:- Circulation:-** If $C$ is a simple closed curve (ie a curve which does not intersect itself anywhere), then the line integral of $\vec{f}$ around $C$ is called the circulation of $\vec{f}$ about $C$ and is denoted by $\oint_C \vec{f}\cdot d\vec{r}$




**Example 5:-** Find the circulation of $\vec{f}$ around the curve $C$, where $\vec{f} = \sin y\,\hat{i} + x(1+\cos y)\hat{j}$ and $C$ is the circle $x^2+y^2=a^2$, $z=0$

**Solution:-**

Circulation of $\vec{f}$ around the curve $C$ is $\oint_C \vec{f}\cdot d\vec{r}$




Here $C: x^2+y^2=a^2$, $z=0$

The Parametric equation of the circle $x^2+y^2=a^2$, $z=0$ are

$x = a\cos\theta$, $y = \sin\theta$, $z=0$ where $0 \le \theta \le 2\pi$

[In case of circular curve, always take Parametric eq. of the given curve, to make the integration easy.]

Now $\vec{r} = x\hat{i} + y\hat{j}$

$d\vec{r} = dx\hat{i} + dy\hat{j}$

$\therefore$

$$\oint_C \vec{f}\cdot d\vec{r} = \oint_C [\sin y\,\hat{i} + x(1+\cos y)\hat{j}]\cdot[dx\,\hat{i} + dy\,\hat{j}]$$

$$= \oint_C [\sin y\,dx + x(1+\cos y)\,dy]$$

$$= \oint_C [\sin y\,dx + x\,dy + x\cos y\,dy]$$

$$= \oint_C \underbrace{[\sin y\,dx + x\cos y\,dy]}_{\Downarrow} + [x\,dy]$$

$$= \oint_C d[x\sin y] + [x\,dy] = \oint_C d[x\sin y] + \oint_C x\,dy$$

$$= \int_0^{2\pi} d[a\cos\theta \cdot \sin(a\sin\theta)] + \int_0^{2\pi} a\cos\theta \cdot a\cos\theta\,d\theta$$

Putting
$x = a\cos\theta$
$y = a\sin\theta$
$dy = a\cos\theta\,d\theta$

$$= [a\cos\theta \sin(a\sin\theta)]_0^{2\pi} + a^2\int_0^{2\pi} \cos^2\theta\,d\theta$$

$$= [a\cos 2\pi \sin(a\sin 2\pi) - a\cos 0 \sin(a\sin 0)] + a^2\int_0^{2\pi} \frac{1+\cos 2\theta}{2}\,d\theta$$

$$= [0-0] + \frac{a^2}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi} = \frac{a^2}{2}[2\pi + 0]$$

$$= \boxed{\pi a^2}$$

Def:- **Work Done by a Force** :- Let $\vec{F}$ be the force acting on a particle moving along a path C. The work done by the force $\vec{F}$ during a small displacement $d\vec{r}$ is given by $\vec{F}\cdot d\vec{r}$

i.e $\quad dW = \vec{F}\cdot d\vec{r} \quad \rightarrow$ small work done.

Total work done by $\vec{F}$ during displacement along C is

$$W = \int_C dW = \int_C \vec{F}\cdot d\vec{r}$$

Scanned by CamScanner

Ex 7) Find the work done in moving a particle in the field of force $\vec{F} = 3x^2\hat{i} + (2xz - y)\hat{j} + z\hat{k}$ along a line joining the points $(0,0,0)$ to $(2,1,3)$.

Solution:-

The equation of a line joining the two points $(0,0,0)$ and $(2,1,3)$ is

$$\frac{x-0}{2-0} = \frac{y-0}{1-0} = \frac{z-0}{3-0} = t \text{ (say)}$$

$$\Rightarrow \frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t \Rightarrow \left.\begin{array}{l} x = 2t \\ y = t \\ z = 3t \end{array}\right]$$

$\therefore x = 2t,\ y = t,\ z = 3t$ are the parametric form of the line.

Here at point $(0,0,0)$, $t = 0$

$(1,2,3)$, $t = 1$.

Here given curve is in $xyz$.
Since line is formed joining $(0,0,0)$ to $(2,1,3)$
$\therefore$ $x$ varies 0 to 2
$y \longrightarrow$ 0 to 1
$z \longrightarrow$ 0 to 3

Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\vec{r} = 2t\hat{i} + t\hat{j} + 3t\hat{k}$

$\frac{d\vec{r}}{dt} = 2\hat{i} + \hat{j} + 3\hat{k}$

Now; in term of $t$,

$$\vec{F} = 3x^2\hat{i} + (2xz - y)\hat{j} + z\hat{k}$$

$$= 3(2t)^2\hat{i} + [2\cdot 2t\cdot 3t - t]\hat{j} + 3t\hat{k}$$

$$= 12t^2\hat{i} + [12t^2 - t]\hat{j} + 3t\hat{k}.$$

Now, total work done is

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot\left(\frac{d\vec{r}}{dt}\right)dt$$

$$= \int_0^1 [12t^2\hat{i} + (12t^2 - t)\hat{j} + 3t\hat{k}]\cdot[2\hat{i} + \hat{j} + 3\hat{k}]\,dt$$

$$= \int_0^1 [24t^2 + (12t^2 - t) + 9t]\,dt$$

$$= \int_0^1 [36t^2 + 8t]\,dt = \left[\frac{36t^3}{3} + \frac{8t^2}{2}\right]_0^1$$

$$= \left[12t^3 + 4t^2\right]_0^1 = \boxed{16}$$

Example 9 If $\phi = 2xyz^2$, $\vec{F} = xy\hat{i} - z\hat{j} + x^2\hat{k}$ and C is the curve $x = t^2$, $y = 2t$, $z = t^3$ from $t = 0$ to $t = 1$, evaluate.

(i) $\int_C \phi\, d\vec{r}$

Sol:- The given curve is $x = t^2$, $y = 2t$, $z = t^3$

$\Rightarrow dx = 2t\,dt$, $dy = 2dt$, $dz = 3t^2 dt$

Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

$= 2t\,dt\,\hat{i} + 2dt\,\hat{j} + 3t^2 dt\,\hat{k}$

$d\vec{r} = [2t\hat{i} + 2\hat{j} + 3t^2\hat{k}]\,dt$

Now, in terms of t ; $\phi$ becomes

$\phi = 2xyz^2$

$= 2 \cdot t^2 \cdot 2t \cdot (t^3)^2 = 4t^9$

Now

$$\int_C \phi\, d\vec{r} = \int_C [4t^9][2t\hat{i} + 2\hat{j} + 3t^2\hat{k}]\,dt$$

$$= \int_0^1 [8t^{10}\hat{i} + 8t^9\hat{j} + 12t^{11}\hat{k}]\,dt$$

$$= \left[\frac{8t^{11}}{11}\hat{i} + \frac{8t^{10}}{10}\hat{j} + 12\frac{t^{12}}{12}\hat{k}\right]_0^1$$

$$= \left[\frac{8}{11}\hat{i} + \frac{8}{10}\hat{j} + \hat{k}\right]$$