1 APRIL 2020 CLASS – B·A/B·SC – III ①

TOPIC – Some thms on I.P.S.

Imp

Thm :- Prove that every finite dimensional vector Space is an Inner Product Space.

Proof :- Let V(F) be a finite dimensional vector Space over a field F (Real or complex). s.t. dim V = n.

Let $S = \{u_1, u_2, \ldots, u_n\}$ be a basis of V. Then every vectors of V can be written as a linear Combination of the elts of S.

Let $u, v, w \in V$ be arbitrary vectors of V.

Then
$$u = \sum_{i=1}^{n} a_i u_i \quad , \quad v = \sum_{i=1}^{n} b_i u_i \quad , \quad w = \sum_{i=1}^{n} c_i u_i$$

Where $a_i, b_i, c_i \in F$.

Define $\langle u, v \rangle = \sum_{i=1}^{n} a_i \bar{b}_i$ — ①

$\because u = a_1 u_1 + a_2 u_2 + \ldots + a_n u_n$
$u = (a_1, a_2, \ldots, a_n)$
$|||^{ly}$ $v = b_1 u_1 + b_2 u_2 + \ldots + b_n u_n$
$v = (b_1, b_2, \ldots, b_n)$

Now, we have to Show that function $\langle , \rangle$ is an I.P.S.

For this we have to Prove 3 Properties of I.P.S.

(i) $\langle u, v \rangle = \overline{\langle v, u \rangle}$

R.H.S ⇒ $\overline{\langle v, u \rangle} = \overline{\sum_{i=1}^{n} b_i \bar{a}_i}$ [By ①]

$$= \sum_{i=1}^{n} \overline{(b_i \bar{a}_i)} = \sum_{i=1}^{n} (\bar{b}_i a_i)$$

$$= \sum_{i=1}^{n} a_i \bar{b}_i = \langle u, v \rangle = \text{L.H.S.}$$

(ii) $\langle u, u \rangle \geq 0$ and $\langle u, u \rangle = 0$ iff $u = 0$

Now $\langle u, u \rangle = \sum_{i=1}^{n} a_i \bar{a}_i$ [By def. ①]

$$= \sum_{i=1}^{n} |a_i|^2 = |a_1|^2 + |a_2|^2 + \ldots + |a_n|^2$$

$$\geq 0$$

$\therefore \langle u, u \rangle \geq 0$

Also $\langle u, u \rangle = 0$ iff $\sum_{i=1}^{n} |a_i|^2 = 0$

$\Leftrightarrow |a_1|^2 + |a_2|^2 + \text{---} + |a_n|^2 = 0$

$\Leftrightarrow a_1 = a_2 = \text{———} = a_n = 0$

$\Leftrightarrow u = 0$

(III) $\langle au + bv, w \rangle = a\langle u, w \rangle + b\langle v, w \rangle$

L.H.S $\Rightarrow$

$$\langle au+bv, w \rangle = \langle \sum_{i=1}^{n} (aa_i + bb_i) u_i, \sum_{i=1}^{n} c_i u_i \rangle$$

$$= \sum_{i=1}^{n} (aa_i + bb_i)\overline{c_i}$$

$\because u = \sum a_i u_i$
$v = \sum b_i u_i$
$au+bv = \sum (aa_i + bb_i) u_i$

$$= \sum_{i=1}^{n} aa_i \overline{c_i} + \sum_{i=1}^{n} bb_i \overline{c_i}$$

$$= a\sum_{i=1}^{n} a_i \overline{c_i} + b\sum_{i=1}^{n} b_i \overline{c_i}$$

$$= a\langle u, w \rangle + b\langle v, w \rangle = R.H.S$$

Hence; all three cond. of I.P.S are satisfied
$\langle , \rangle$ is an I.P.S. on $V(F)$.

$\therefore$ every finite dimensional vector space $V(F)$ is an I.P.S

IMP

Thm :— Let f be a linear functional on a finite dimensional I.P.S. $V(F)$. Then $\exists$ a unique vector $v \in V$ such that $f(u) = \langle u, v \rangle$ for all $u \in V$.

PF:-

[Linear Functional :— f is s.t.b. linear functional on $V(F)$ if

$\exists$ a map $f : V \to F$ s.t

$f(\alpha u + \beta v) = \alpha f(u) + \beta f(v)$; where $u, v \in V$
$\alpha, \beta \in F$]

Let $f: V \to F$ be a linear functional on V.
Since, V is a finite dimensional I.P.S. over F. Let $\dim V = n$.
Consider $S = \{u_1, u_2, \ldots, u_n\}$ be an orthonormal basis of finite dim. I.P.S. $V(F)$ so that
(i) $\langle u_i, u_j \rangle = 1$ iff $i = j$
(ii) $\langle u_i, u_j \rangle = 0$ iff $i \neq j$
[By def. of orthonormal set]

Define vector v as
$$v = \sum_{i=1}^{n} \overline{f(u_i)}\, u_i \quad (1)$$
[$\because u_i \in V$ we can define $f(u_i) = \langle u_i, v \rangle$ As given in the statement]

Here; clearly $v \in V$.

Define a map $g: V \to F$ such that
$$g(\alpha) = \langle \alpha, v \rangle \text{ for all } \alpha \in V \quad (2)$$

<u>Claim</u>:- g is a linear functional.
Let $a, b \in F$ and $\alpha, \beta \in V$
Then $g(a\alpha + b\beta) = \langle a\alpha + b\beta, v \rangle$ [By (2)]
$= a\langle \alpha, v \rangle + b\langle \beta, v \rangle$

$g(a\alpha + b\beta) = a\,g(\alpha) + b\,g(\beta)$

$\Rightarrow g$ is a Linear Functional.

For $1 \le j \le n$, we have $g(u_j) = \langle u_j, v \rangle$ [By (2)]
$$= \langle u_j, \underbrace{\sum_{i=1}^{n} \overline{f(u_i)}\, u_i}_{\alpha} \rangle \quad \text{[By (1)]}$$
$$= \overline{\sum_{i=1}^{n} \overline{f(u_i)}}\, \langle u_j, u_i \rangle$$
$$= \sum_{i=1}^{n} f(u_i) \langle u_j, u_i \rangle$$
$$= f(u_j)$$
[$\because$ When $i = j$ $\langle u_i, u_j \rangle = 1$; $i \neq j$ $\langle u_i, u_j \rangle = 0$]

(4)

$\therefore\ g(u_j) = f(u_j) \quad ; \; u_j \in S$

$\Rightarrow$ f and g are both equal on $S \subseteq V$

$\Rightarrow$ f and g are both equal on V. i.e $f = g$

$\therefore$ for every $u \in V$, $\exists$ a vector $v \in V$ such that $f(u) = g(u) = \langle u, v \rangle$ for all $u \in V$. [By (2)]

Uniqueness :- R.T.P $v \in V$ is unique.

If possible, let $w \in V$ be any other vector such that $f(u) = \langle u, w \rangle$ for all $u \in V$

ALSO $f(u) = \langle u, v \rangle$ for all $u \in V$ [given]

$\therefore\ \langle u, w \rangle = \langle u, v \rangle$

$\Rightarrow \langle u, w \rangle - \langle u, v \rangle = 0$

$\Rightarrow \langle u, w - v \rangle = 0$

$\Rightarrow w - v = 0 \Rightarrow \boxed{w = v}$

In particular Take $u = w - v \in V$. $\langle w-v, w-v \rangle = 0$ $\Rightarrow w - v = 0$

This proves the uniqueness.