Some Important Questions based on Vector Calculus

[H-1]

Q1.) If $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0$ and the vectors

$$\vec{A} = \hat{i} + a\hat{j} + a^2\hat{k}$$
$$\vec{B} = \hat{i} + b\hat{j} + b^2\hat{k}$$
$$\vec{C} = \hat{i} + c\hat{j} + c^2\hat{k}$$

are non-coplanar, then prove that $abc = -1$.

Q2) Find the volume of the parallelopiped whose conterminous edges are represented by the vectors $2\hat{i} - 3\hat{j} + 4\hat{k}$, $\hat{i} + 2\hat{j} - \hat{k}$, $3\hat{i} - \hat{j} + 2\hat{k}$.

Q3). For what value of 'λ', the vectors $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} - 3\hat{k}$, $\vec{c} = 3\hat{i} + \lambda\hat{j} + 5\hat{k}$ are coplanar.

Q4) Show that $[\vec{a}+\vec{b} \quad \vec{b}+\vec{c} \quad \vec{c}+\vec{a}] = 2[\vec{a} \; \vec{b} \; \vec{c}]$

Q5) Show that

$$[\vec{l} \; \vec{m} \; \vec{n}][\vec{a} \; \vec{b} \; \vec{c}] = \begin{vmatrix} \vec{l}\cdot\vec{a} & \vec{l}\cdot\vec{b} & \vec{l}\cdot\vec{c} \\ \vec{m}\cdot\vec{a} & \vec{m}\cdot\vec{b} & \vec{m}\cdot\vec{c} \\ \vec{n}\cdot\vec{a} & \vec{n}\cdot\vec{b} & \vec{n}\cdot\vec{c} \end{vmatrix}$$

Q6) If $\vec{a}, \vec{b}, \vec{c}$ be three unit vectors such that $\vec{a} \times (\vec{b} \times \vec{c}) = \frac{1}{2}\vec{b}$, Find the angles which $\vec{a}$ makes with $\vec{b}$ and $\vec{c}$ where $\vec{b}$ and $\vec{c}$ are non-parallel.

Q7) Prove that

$$[\vec{a}\times\vec{b} \quad \vec{b}\times\vec{c} \quad \vec{c}\times\vec{a}] = \begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c} \\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c} \\ \vec{c}\cdot\vec{a} & \vec{c}\cdot\vec{b} & \vec{c}\cdot\vec{c} \end{vmatrix}$$
$$= [\vec{a} \; \vec{b} \; \vec{c}]^2$$

Q8) Prove that $(\vec{b}\times\vec{c})\cdot(\vec{a}\times\vec{d}) + (\vec{c}\times\vec{a})\cdot(\vec{b}\times\vec{d}) + (\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = 0$

Q9) Find the set of vectors reciprocal to $\vec{a}, \vec{b}, \vec{a}\times\vec{b}$

[H-2]

Q10) Let $\vec{a} = \vec{p}\cos u + \vec{q}\sin u$ where $\vec{p}, \vec{q}$ are constant vectors. Prove that $\vec{a}\cdot\left(\frac{d\vec{a}}{du} \times \frac{d^2\vec{a}}{du^2}\right) = 0$

Q11) Evaluate $\frac{d}{dt}\left[\vec{r}\times\left(\frac{d\vec{r}}{dt}\times\frac{d^2\vec{r}}{dt^2}\right)\right]$

Q12) If $\vec{a}=\sin\theta\,\hat{i}+\cos\theta\,\hat{j}+\theta\hat{k}$, $\vec{b}=\cos\theta\,\hat{i}-\sin\theta\,\hat{j}-3\hat{k}$ and $\vec{c}=2\hat{i}+3\hat{j}-3\hat{k}$. Find $\frac{d}{d\theta}\{\vec{a}\times(\vec{b}\times\vec{c})\}$ at $\theta=\frac{\pi}{2}$

Q13) A particle moves along the curve $x=3t^2, y=t^2-2t, z=t^3$. Find the velocity and acceleration at $t=1$

Q14) A particle moves along the curve $x=4\cos t$, $y=4\sin t$, $z=6t$, Find velocity and acceleration at $t=0$ and $t=\pi/2$

Q15) Show that if $\vec{u},\vec{v},\vec{w}$ are constant vectors, then $\vec{r}=\vec{u}t^2+\vec{v}t+\vec{w}$ is the path of a particle moving with constant acceleration.

CH-3

Q16) If $\vec{a}=x^2yz\,\hat{i}-2xz^3\hat{j}+xz^2\hat{k}$ and $\vec{b}=2z\hat{i}+y\hat{j}-x^2\hat{k}$ Find $\frac{\partial^2}{\partial x\partial y}(\vec{a}\times\vec{b})$ at the point $(1,0,-2)$.

Q17) Prove that $\vec{a}\cdot\nabla\left(\vec{b}\cdot\nabla\frac{1}{r}\right)=\frac{3(\vec{a}\cdot\vec{r})(\vec{b}\cdot\vec{r})}{r^5}-\frac{\vec{a}\cdot\vec{b}}{r^3}$

Q18) Prove that $\operatorname{div}(\operatorname{grad} r^n)=n(n+1)r^{n-2}$

Q19) Determine the constant 'a' so that the vector $\vec{f}=(x+3y)\hat{i}+(y-2z)\hat{j}+(x+az)\hat{k}$ is solenoidal.

Q20) Show that $\operatorname{div}\left[\frac{f(r)\,\vec{r}}{r}\right]=\frac{1}{r^2}\frac{d}{dr}\left(r^2 f(r)\right)$

Where $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$ and $r=|\vec{r}|$.

Q21) Prove that a constant vector is irrotational but converse is Not true. give example.

Q22) Let $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$ and $r=|\vec{r}|$ then Show that $\operatorname{curl}(f(r)\vec{r})=0$

Q23) If $\vec{f}$ and $\vec{g}$ are irrotational, Prove that $\vec{f}\times\vec{g}$ is solenoidal.

Q24) If $\vec{a}$ is constant vector and $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$ Find $\operatorname{curl}[(\vec{a}\times\vec{r})r^n]$, where $r=|\vec{r}|$.

Q25) If $\vec{a}$ is a constant vector and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ then prove that $curl\left(\frac{\vec{a}\times\vec{r}}{r^3}\right) = \frac{-\vec{a}}{r^3} + \frac{3\vec{r}}{r^5}(\vec{a}\cdot\vec{r})$

Q26) Prove that the function $\frac{1}{r}$ $(r \neq 0)$ where

$r = \sqrt{x^2+y^2+z^2}$ is harmonic.

Q27) Find the directional derivative of $\phi = x^2 - y^2 + 2z^2$ at the point $P(1,2,3)$ in a direction towards the point $Q(5,0,4)$

Q28) Find the directional derivative of $\phi(x,y,z) = x^2yz + 4xz^2$ at the point $(1,-2,1)$ in the direction $2\hat{i} - \hat{j} - 2\hat{k}$.

Q29) Given the curve of intersection of two surfaces $x^2+y^2+z^2 = 1$ and $x+y+z=1$; Find the tangent line at the point $(1,0,0)$.

Ch-4

Q30) If $u = 2x+3$, $v = y-4$, $w = z+2$, Show that u, v, w are orthogonal.

Q31) Prove that cylinderical co-ordinate system is self-reciprocal.

Q32) If we denote cylinderical co-ordinates by $x = r\cos\theta$, $y = r\sin\theta$, $z = z$ find ∇r and $\nabla\theta$.

Q33) Transform the function $\vec{f} = 3ar^2\sin\theta\cos\phi\,\hat{e}_r + 2a^2r\cos\theta\sin\phi\,\hat{e}_\theta + r^3\hat{e}_\phi$ in cartesian co-ordinates.

Q34) If $\psi = xyz$, evaluate $\nabla\psi$ in spherical coordinates.

Q35) Express the velocity and acceleration of a particle in spherical co-ordinates.

CH-5

Q36) If $\vec{r} = 5t^2\hat{i} + t\hat{j} - t^3\hat{k}$ evaluate $\int_1^2 \vec{r} \times \frac{d^2\vec{r}}{dt^2}\, dt$

Q37) Find the circulation of $\vec{F}$ around the curve C where $\vec{F} = \sin y\,\hat{i} + x(1+\cos y)\hat{j}$ and C is the circle $x^2+y^2=a^2$, $z=0$

Q38) Evaluate $\iint_S \vec{F}\cdot\hat{n}\, dS$, where $\vec{F} = yz\hat{i} + zx\hat{j} + xy\hat{k}$

and S is the part of the surface of the sphere $x^2+y^2+z^2=1$ which lies in the first octant.

Q39) If $\vec{F} = (2x^2-3z)\hat{i} - 2xy\hat{j} - 4x\hat{k}$, evaluate $\iiint_V \nabla\times\vec{F}\, dV$ where V is the region bounded by the planes $x=0, y=0, z=0$ and $2x+2y+z=4$.

Q40) If $\vec{F} = \phi A$, $A = \nabla\phi$ and $\nabla^2\phi = 0$, then show that $\iint_S \vec{F}\cdot\hat{n}\, dS = \iiint_V A^2\, dV$

Q41) Evaluate $\iint_S (y^2z^2\hat{i} + z^2x^2\hat{j} + z^2y^2\hat{k})\cdot\hat{n}\, dS$,

where S is the part of the sphere $x^2+y^2+z^2=1$ above the xy-plane and bounded by xy-plane.

Q42) Show that the area bounded by a simple closed curve C is given by $\frac{1}{2}\oint_C (x\,dy - y\,dx)$.

Hence find the area of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.