Class – B.Sc. | B.A. (2nd Sem)

Subject – Number Theory and Trigonometry

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UNIT-I

Q-1 Prove that product of any r consecutive integers is divisible by $r!$

Q-2 Prove that an integer is divisible by 9 iff sum of its digits is divisible by 9.

Q-3 Prove that a non-zero integer is not an exact divisor of any number.

Q-4 For any two integers a and b where $b>0$, prove that there exist unique numbers q and r s.t. $a = bq + r$ where $0 \le r < b$.

Q-5 Use mathematical induction, prove that $9 \mid (10^n + 3 \cdot 4^{n+2} + 5)$.

Q-6 State & prove Gauss theorem.

Q-7 Find (i) (858, 325) (ii) (592, 252). Also express g.c.d as linear combination of numbers.

Q-8 Prove that the product of two positive integers is equal to the product of their L.C.M and G.C.D.

Q-9 State & prove Euclid's first theorem.

Q-10 Show that there are infinitely many primes of the form (i) $4n+3$.
(ii) $6n+5$.

Q-11 Prove that ${}^nC_r$ is an integer.

Q-12 Find remainder obtained on dividing
(i) 3^{181} by 17. (ii) 2^{20} by 7.

Q-13 Solve (i) $259x \equiv 5 \pmod{11}$
(ii) $15x \equiv 10 \bmod (145)$
(iii) $13x \equiv 9 \bmod (25)$
(iv) $7x \equiv 5 \bmod (256)$

Q-14 Find all solutions in positive integers of $5x + 3y = 52$

Q-15 Find least positive solutions of
(i) $11x + 5y = 79$ (ii) $436x - 393y = 5$

Q-16 State & prove Fermat's theorem

Q-17 Show that $n^{16} - a^{16}$ is divisible by 85 if n and a are co-prime to 85.

Q-18 State & prove Wilson's theorem.

Q-19 Find remainder when $2(26)!$ is divisible by 29.

Q-20 State & prove Chinese Remainder theorem.

Q-21 Find all integers that give remainder 1, 2, 3 when divided by 3, 4, 5 respectively.

Q-22 If p is a prime number, show that $(p-2r)!\,(2r-1)!-1$ is divisible by p.

UNIT-II

Q-23 If $n>1$ has prime factorization $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$ then $\phi(n) = n\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\cdots\left(1-\frac{1}{p_r}\right)$

Q-24 For $n>2$, prove $\phi(n)$ is an integer.

Q-25 Find (i) $\phi(600)$ (ii) $\phi(462)$

Q-26 Show that $\phi(n) = \phi(n+1) = \phi(n+2)$ for $n = 5186$

Q-27 Prove that $\phi(n) = \frac{n}{2}$ iff $n = 2^k$ for some integer $k \geq 1$

Q-28 Let a be any integer and m be a positive integer. If $(a, m) = 1$ then $a^{\phi(m)} \equiv 1 \pmod{m}$

Q-29 Show that $2, 4, 6 \cdots 2m$ is CRS (mod m) if m is odd.

Q-30 Find highest power of 9 which divides 365!

Q-31 State & prove Euler's Generalization of Fermat's theorem

Q-32 Evaluate $d(p^2q^3)$ where p, q are distinct primes. Also, find σ-function of the number.

Q-33 Solve the equation $x^4+x^3+x^2+x+1=0$

Q-34 Find all n such that $d(n)=10$. Hence find least such value of n.

Q-35 Prove that $\mu(n)\,\mu(n+1)\,\mu(n+2)\,\mu(n+3)=0$ if n is positive integer.

Q-36 Show that the set of integers $\{1, 5, 7, 11\}$ is RRS (mod 12)

Q-37 Show that 3 is quadratic non-residue of 31

Q-38 Evaluate $\left(\frac{-168}{11}\right)$

Q-39 Is congruence $x^2 \equiv 150 \pmod{1009}$ solvable?

Q-40 Find all primes p such that $\left(\frac{5}{p}\right) = -1$

UNIT- III

Q-41 State and prove De-Moivre's theorem.

Q-42 If $2\cos\alpha = x+\frac{1}{x}$, $2\cos\beta = y+\frac{1}{y}$; show that one of values of $x^m y^n + \frac{1}{x^m y^n}$ is $2\cos(m\alpha + n\beta)$.

Q-43 Form an equation whose roots are $\cos\frac{2\pi}{7}$, $\cos\frac{4\pi}{7}$, $\cos\frac{8\pi}{7}$. Hence form an equation whose roots are $\sec\frac{2\pi}{7}$, $\sec\frac{4\pi}{7}$, $\sec\frac{8\pi}{7}$

Q-44 If $x_r = \cos\frac{\pi}{3^r} + i\sin\frac{\pi}{3^r}$, show that $x_1 \cdot x_2 \cdot x_3 \cdots x_n = \cos\left[\frac{\pi}{2}\left(1-\frac{1}{3^n}\right)\right] + i\sin\left[\frac{\pi}{2}\left(1-\frac{1}{3^n}\right)\right]$

Q-45 Prove that 'n' n^{th} roots of unity form a series of G.P. and show that their product is $(-1)^{n-1}$.

Q-46 Show that roots of $(1+x)^{2n} + (1-x)^{2n} = 0$ are given by, $\pm i\tan\frac{(2r-1)\pi}{4n}$ where $r = 1, 2, 3, \ldots, n$.

Q-47 Prove that $\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$.

Q-48 Prove $\cos^5\theta\cos^3\theta = \frac{-1}{27}(\sin 8\theta + 2\sin 6\theta - 2\sin 4\theta - 6\sin 2\theta)$.

Q-49 Express $\cos^6\theta$ in terms of cosines of multiples of θ.

Q-50 If α and β are imaginary cube roots of unity then prove $\alpha\cdot e^{\alpha x} + \beta\cdot e^{\beta x} = -e^{-\frac{x}{2}}\left[\sqrt{3}\sin\frac{\sqrt{3}}{2}x + \cos\frac{\sqrt{3}}{2}x\right]$.

Q-51 Prove that $\tanh 2x = \frac{2\tanh x}{1+\tanh^2 x}$.

Q-52 Separate into real and imaginary parts:-

(a) $\tanh(x+iy)$ b) $\operatorname{cosec}(x+iy)$

Q-53 If $\tan(\theta+i\phi) = \cos\alpha + i\sin\alpha$ then prove $e^{2\phi} = \tan\left(\frac{\pi}{2}+\frac{\alpha}{2}\right)$

Q-54 If $z = x+iy$, where x and y are real, find the real and imaginary parts of $\frac{\cos z}{z+1}$.

Q-55 If $\tan(A+iB) = x+iy$, prove that $x^2+y^2-2y\coth(2B) = -1$

UNIT -IV

Q-56 Resolve $\log(1+i)$ into real and imaginary part.

Q-57 Express $\log[\operatorname{Log}(\cos\theta + i\sin\theta)]$ in form $A+iB$.

Q-58 Seperate $\log(\sin(x+iy))$ into real and imaginary parts.

Q-59 Prove that $(i^i)^i = \cos\theta - i\sin\theta$ where $\theta = (4n+1)\frac{\pi}{2}$

Q-60 Prove that $\log_i 2 = \dfrac{\frac{1}{2}\log 2 + i n\pi}{\log 2 + i n\pi}$

Q-61 Prove that $\log(-5) = \log 5 + i(2n+1)\pi$

Q-62 Prove that $\tanh\frac{u}{2} = \tan\frac{\theta}{2}$, if $u = \log\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$

Q-63 Prove that $i^i = e^{-(4n+1)\frac{\pi}{2}}$ and show that its values form a geometrical progression.

Q-64 If $i^{\alpha+i\beta} = a+ib$, prove that $a^2+b^2 = e^{-(4n+1)\pi\beta}$

Q-65 Prove that $2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}$.

Q-66 Prove $\sin^{-1}\frac{4}{5} + \sin^{-1}\frac{5}{13} + \sin^{-1}\frac{16}{25} = \frac{\pi}{2}$.

Q-67 If $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$, show $x^2+y^2+z^2+2xyz=1$

Q-68 Solve equation $\tan^{-1}2x + \tan^{-1}3x = \frac{\pi}{4}$

Q-69 Separate $\tan^{-1}(x+iy)$ into real and imaginary parts.

Q-70 Prove $\log\tan\left(\frac{\pi}{4} + \frac{x}{2}i\right) = i\tan^{-1}(\sinh x)$.

Q-71 Find sum of series $\sin x + \sin 3x + \cdots$ to n terms and deduce sum of series $1+3+5+\cdots+(2n-1)$.

Q-72 Solve the equation:

$$\tan^{-1}\frac{1}{4} + 2\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{6} + \tan^{-1}\frac{1}{x} = \frac{\pi}{4}$$

Q-73 Prove that

$$\sinh^{-1}x = n\pi i + (-1)^n\log\left[x + \sqrt{x^2+1}\right]$$

Q-74 Sum to n terms the series

$$\cot^{-1}(2\cdot 1^2) + \cot^{-1}(2\cdot 2^2) + \cot^{-1}(2\cdot 3^2) + \cdots$$

Q-75 Find the real and imaginary parts of $\tanh^{-1}(x+iy)$.

Q-76 Find the sum of the series $\sin\alpha + \frac{1}{2}\sin 2\alpha + \left(\frac{1}{2}\right)^2\sin 3\alpha + \cdots$ to ∞.

Q-77 Find the real part of the principal value of $\log(1+e^{i2\theta})$.

Q-78 Find sum of series ~~Sinx+~~ $\sec\alpha\sec 2\alpha + \sec 2\alpha\sec 3\alpha + \cdots$ to n terms

Q-79 Sum the series $\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} + \cdots$ to n terms & deduce the sum to infinity.

Q-80 Find sum to infinity of series

$$1 - \frac{1}{2}\cos\theta + \frac{1\cdot 3}{2\cdot 4}\cos 2\theta - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cos 3\theta + \cdots), \quad -\pi < \theta < \pi$$