SECTION ~ IV

Defn:- Separation of Variables:-

The method of separation of variables tries to find out the solution of given PDE by taking solution as a combination of functions of different variables

Example:- Solution of heat eqn, solution of hamilton Jacobi PDE

Defn:- Eigen Value:- Let $U \subseteq \mathbb{R}^n$ be open and bounded. We say that any parameter λ is eigen value of the operator $-\Delta$ on ∂U if $\exists$ a function $w \neq 0$ satisfying the equation

$$-\Delta w = \lambda w \quad \text{in } U$$

$$w = 0 \quad \text{on } \partial U$$

$\boxed{Ax = \lambda x}$

Also the function w is called Eigen function corresponding to eigen value λ.

Solution of heat eqn by method of separation of variable:-

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Statement:- Let $U \subseteq \mathbb{R}^n$ be open and bounded.

Consider heat equation (PDE)

$$\left.\begin{aligned} u_t - \Delta u &= 0 \quad \text{in } U \times (0,\infty) \\ u &= 0 \quad \text{on } \partial U \times [0,\infty) \\ u &= g \quad \text{on } U \times \{t=0\} \end{aligned}\right\} \quad \text{—— (1)}$$

where $g : U \to \mathbb{R}$ is a known function.

Find the solution of eqn (1) by method of separation of variable.

Soln:- Let us consider $u = u(x,t)$ be a solution of eqn (1) for $x \in U, t \geq 0$

Let $u(x,t) = v(t) w(x)$ —— (2)

Then $u_t = v_t(t) w(x) = v'(t) w(x)$

and $\Delta u = v(t) \Delta w(x)$ (Δ operate only fun^n of x) $\quad \Delta \equiv D_x^2$

Now $u(x,t)$ is a soln of eqn (1) iff $u_t - \Delta u = 0$

$$\Leftrightarrow v_t w - v \Delta w = 0$$

$$\Leftrightarrow v_t w = v \Delta w$$

$$\Leftrightarrow \frac{v_t}{v} = \frac{\Delta w}{w}$$

$$\text{i.e., } \frac{v_t(t)}{v(t)} = \frac{\Delta w(x)}{w(x)}$$

Now L.H.S. is a function of t and R.H.S. is a function of x

$\therefore$ Above eqn is possible only if both are equal to a constant.

$$\Rightarrow \frac{v_t(t)}{v(t)} = \frac{\Delta w(x)}{w(x)} = \mu \text{ (say)}$$

Now, $\frac{v_t(t)}{v(t)} = \mu$

Integrating w.r.t. t, we have

$\log v(t) = \mu t + A$

$\Rightarrow v(t) = e^{\mu t + A}$

$= e^{\mu t} \cdot e^{A}$

$= e^{\mu t} b$ where $b = e^{A}$

i.e., $v(t) = b e^{\mu t}$ ——— (3)

Also, $\frac{\Delta w(x)}{w(x)} = \mu$

$\Rightarrow \Delta w(x) = \mu w(x)$ ——— (4)

Let us consider λ is eigen value of the operator $-\Delta$, then by defn $\exists$ a function w such that

Learn

$$-\Delta w = \lambda w \quad \text{in } U$$
$$w = 0 \quad \text{on } \partial U$$

Set $\mu = -\lambda$ in eqn (4), we get

$$\Delta w = -\lambda w$$

and also $w = 0$ on ∂U

$\because u = 0$ on $\partial U \times [0, \infty)$ (By eqn (1))

$\Rightarrow w(x) v(t) = 0$ on $\partial U \times [0, \infty)$ ↳ zero nhi ho skti (from eqn (3))

$\Rightarrow w(x) = 0$ on ∂U → Isme t nhi h too hmm $[0,\infty)$ nhi likhe

↳ In eqn (3), exponential kbi zero nhi ho skta

Hence $v(t) = b e^{\mu t}$

$= b e^{-\lambda t}$

and $u(x,t) = w(x) v(t)$

$= w(x) b e^{-\lambda t}$

satisfies the boundary value problem

$$u_t - \Delta u = 0 \quad \text{in } U \times (0, \infty)$$
$$u = 0 \quad \text{on } \partial U \times [0, \infty)$$

with condition $u(x, 0) = b w(x)$

$$\left[\begin{array}{l} u(x,t) = w(x) b e^{-\lambda t} \\ \text{when } t = 0 \quad u(x,0) = w(x) b \end{array}\right]$$

Hence $u(x,t)$ is the solution of boundary value problem

$$\left.\begin{aligned} &u_t - \Delta u = 0 && \text{in } U \times (0,\infty) \\ &u = 0 && \text{on } \partial U \times [0,\infty) \\ &u = g && \text{on } U \times \{t=0\} \end{aligned}\right\} \quad -(5)$$

only if $g(x) = b\omega(x)$

$\left\{ \begin{array}{l} u(x,0) = b\omega(x) \\ \text{and } u = g \text{ This is} \\ \text{possible only } g(x) = b\omega(x) \end{array} \right.$

Hence $u(x,t) = b\omega(x)e^{-\lambda t}$

is a solⁿ of eqⁿ (5) under some conditions

(i) $g(x) = b\omega(x)$

(ii) λ is eigen value for operator $-\Delta$.

(iii) ω is solution of boundary value problem

$$\Delta\omega = -\lambda\omega \quad \text{in } U$$
$$\omega = 0 \quad \text{on } \partial U$$

Similarity Solutions

Defⁿ:- Travelling Wave :- A solution $u = u(x,t) = v(x-\sigma t)$; $x \in \mathbb{R}, t \in \mathbb{R}^+$ of the PDE of two variables x, t is called travelling wave and v is the profile of the wave with speed σ.

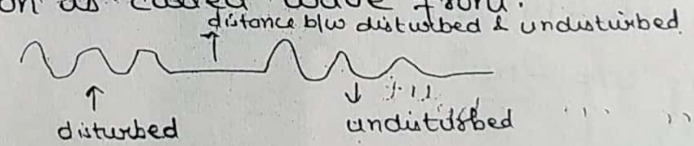
Compulsory

Defⁿ:- Plane Wave :- A solution of the form

$$u = u(x,t) = v(y \cdot x - \sigma t) \; ; \; x \in \mathbb{R}^n, t \in \mathbb{R}^+$$ of PDE

in n+1 variables is called plane wave with wavefront normal to y and velocity $\dfrac{\sigma}{|y|}$

Defⁿ:- Wave front :- Surface between disturbed and undisturbed Region is called wave front.



Defⁿ:- Wave Number :- The number of vibration in a unit distance is called wave number.

generated by wave

Defⁿ:- Frequency :- The number of vibration in unit time is called frequency.

Defⁿ:- Exponential solution of wave :- A solution of the form $u = u(x,t) = e^{i(y \cdot x + \omega t)}$ is called exponential solution for $y \in \mathbb{R}^n$ and $\omega \in$ [illegible] both [illegible]

$u(x,t) = e^{i(y \cdot x - \sigma t)}$

Exponential solution of various PDE:-

$u_t - \Delta u = 0$ —— ①

[I] **Heat Equation:-** Consider a heat equation ① of the exponential form.

Let $u(x,t) = e^{i(y\cdot x + \omega t)}$ be a solution of eqn ①

$y \cdot x = y_1 x_1 + y_2 x_2 + \dots$, Diff. w.r.t. x_i then we get y_i

Now $\dfrac{\partial u}{\partial x_i} = i y_i e^{i(y\cdot x+\omega t)}$

$$\frac{\partial^2 u}{\partial x_i^2} = (iy_i)^2 \underbrace{e^{i(y\cdot x+\omega t)}}_{u(x,t)=u}$$

$$\frac{\partial^2 u}{\partial x_i^2} = -y_i^2 u$$

$$\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = \sum_{i=1}^{n} -y_i^2 u$$

$$\Rightarrow \Delta u = -u \sum_{i=1}^{n} y_i^2 \qquad \left[\because \Delta = \nabla^2 = D^2 = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}\right]$$

$$\Rightarrow \Delta u = -u|y|^2$$

$y = (y_1, y_2, \dots y_n)$
$|y| = \sqrt{y_1^2 + y_2^2 + \dots + y_n^2}$
$y^2 = y_1^2 + y_2^2 + \dots + y_n^2 = \sum y_i^2$

and $u_t = i\omega e^{i(y\cdot x+\omega t)}$

$$u_t = i\omega u$$

Put both these value in eqn ①

$$i\omega u - (-u|y|^2) = 0$$

$$\Rightarrow i\omega u + u|y|^2 = 0$$

$$\Rightarrow u(i\omega + |y|^2) = 0$$

$$\Rightarrow i\omega + |y|^2 = 0 \qquad [u \neq 0 \text{ As we take } u=0 \text{ then } \exists \text{ not sol}^n \text{ of eqn ①}]$$

$$\Rightarrow \omega = \frac{-|y|^2}{i}$$

$$\Rightarrow \omega = i|y|^2$$

$\Rightarrow$ solution of eqn ①. is given by

$$u(x,t) = e^{i(y\cdot x+\omega t)}$$
$$= e^{i(y\cdot x + i|y|^2 t)}$$
$$= e^{iy\cdot x} e^{i^2|y|^2 t}$$
$$= e^{iy\cdot x} e^{-|y|^2 t}$$

remove these bracket

$= e^{-|y|^2 t}(\cos(y\cdot x) + i\sin(y\cdot x))$

→ As time increase then this value decreases

This solution decreases with time and correspond to dissipation

→ Here the power of this value is −ve so dissipation occurs

dissipation: Losses due to friction & any other region

with inc. of time, the density of heat eq decrease

[] Wave Equation:-

Consider a wave equation, $u_{tt} - \Delta u = 0$ —— ①

Let $u(x,t) = e^{i(y\cdot x+\omega t)}$ be a exponential solution of eq^n ①

Now $\dfrac{\partial u}{\partial x_i} = i y_i e^{i(y\cdot x+\omega t)}$

$$\frac{\partial^2 u}{\partial x_i^2} = (iy_i)^2 e^{i(y\cdot x+\omega t)} \Rightarrow \frac{\partial^2 u}{\partial x_i^2} = -y_i^2 u$$

$$\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = \sum_{i=1}^{n} -y_i^2 u$$

$$\Rightarrow \Delta u = -u \sum_{i=1}^{n} y_i^2$$

$$\Rightarrow \Delta u = -u|y|^2$$

and $u_t = i\omega e^{i(y\cdot x+\omega t)}$

$$u_{tt} = (i\omega)^2 e^{i(y\cdot x+\omega t)}$$

$$u_{tt} = -\omega^2 u$$

Now u is solution of eq^n ①

$$\therefore -\omega^2 u - (-u|y|^2) = 0$$

$$\Rightarrow -\omega^2 u + u|y|^2 = 0$$

$$\Rightarrow u(-\omega^2 + |y|^2) = 0$$

$$\Rightarrow -\omega^2 + |y|^2 = 0$$

$$\Rightarrow -\omega^2 = -|y|^2$$

$$\Rightarrow \omega^2 = |y|^2$$

$$\Rightarrow \omega = \pm |y|$$

Hence solution of eq^n ① is $u(x,t) = e^{i(y\cdot x+\omega t)}$

$$= e^{i(y\cdot x \pm |y|t)}$$

$$= e^{iy\cdot x} e^{\pm i|y|t}$$

$$= (\cos y\cdot x + i\sin y\cdot x)(\cos|y|t \pm i\sin|y|t)$$

Here the dissipation depend on $\pm$ sign.

[III] Schrodinger's Equation:-

Consider a schrodinger eq^n $iu_t + \Delta u = 0$ —— ① in $\mathbb{R}^n \times (0,\infty)$

Let $u(x,t) = e^{i(y.x+wt)}$ be a solution of eqn ①

Now $\Delta u = -u|y|^2$ (Prove it) (same as I, II part)

and $u_t = iwu$

Now u is solution of eqn ①

$\therefore i(iwu) + (-u|y|^2) = 0$

$\Rightarrow -wu - u|y|^2 = 0$

$\Rightarrow -u(w + |y|^2) = 0$

$\Rightarrow w + |y|^2 = 0$

$\Rightarrow w = -|y|^2$

Now solution of eqn ① is $u(x,t) = e^{i(y.x+wt)}$

$= e^{i(y.x-|y|^2t)}$

$= \cos(y.x-|y|^2t) + i\sin(y.x-|y|^2t)$

$= e^{iy.x}\, e^{-i|y|^2t}$

$= e^{-i|y|^2t}(\cos(y.x) + i\sin(y.x))$

which correspond to dissipation with time.

[IV] Dispersive or Airy Equation:-

Consider a dispersive eqn $u_t + u_{xxx} = 0$ in $\mathbb{R}\times(0,\infty)$ —— ①

Let $u(x,t) = e^{i(y.x+wt)}$ be an exponential solution of eqn ①

Defn:- Dispersive eqn :- An eqn of the form $u_t + \nabla^3 u = 0$ in $\mathbb{R}^n\times(0,\infty)$ is called dispertive eqn.

Now $u_x = iy\,e^{i(y.x+wt)}$

$u_{xx} = (iy)^2 e^{i(y.x+wt)}$

$u_{xxx} = (iy)^3 e^{i(y.x+wt)}$

$u_{xxx} = -iy^3 u$

and $u_t = iwe^{i(y.x+wt)}$

$u_t = iwu$

Now u is solution of eqn ①

$\therefore iwu + (-iy^3u) = 0$

$\Rightarrow iu(w - y^3) = 0$ (u being a soln doesn't vanishes)

$\Rightarrow w - y^3 = 0$

$\Rightarrow w = y^3$

Now solution of eqn ① is $u(x,t) = e^{i(y.x+wt)}$

$= e^{i(y.x+y^3t)}$

$= e^{iy.x}e^{iy^3t}$

↳ Here the power of this value is +ve so no dissipation occurs.

$= \cos(y.x+y^3t) + i\sin(y.x+y^3t)$

Also $u(x,t) = e^{iy^3t} e^{iyx} = e^{iy^3t}(\cos yx + i \sin yx)$

which correspond no dissipation with time and [have velocity $= \frac{\sigma}{|y|}$

Comparing $(y \cdot x + y^3 t)$ with $v(y \cdot x - \sigma t)$ then we get $\sigma = y^3$

$$\text{Velocity} = \frac{y^3}{|y|} = \frac{|y|^2 y}{|y|} = y|y| = y^2 \text{ (if } y \text{ is +ve)}]$$

Solitones :-

Def^n :- The solution of korteweg-de-vries equation is called solitones

The solution of this equation may be either in secant hyperbolic or cos harp hyperbolic functions.

$u = u(x,t) = A \operatorname{sech}(s+B)$ where $s = x - \sigma t$

Let us consider korteweg-de vries eq^n $\boxed{\text{Put } \sigma = C}$

$$u_t + 6uu_x + u_{xxx} = 0 \quad \text{in } \mathbb{R} \times (0,\infty) \quad \text{—— (1)}$$

we try to find out the sol^n of eq^n (1) of the form $u = u(x,t) = v(s)$ where $s = x - \sigma t$

Let us consider sol^n of eq^n (1) is

$$u = u(x,t) = v(x - \sigma t) \quad \text{—— (2)} \quad x \in \mathbb{R}, t > 0$$

Now $u_t = v'(x - \sigma t)(-\sigma)$ $\quad \left(\because u_t = \frac{dv}{ds} \frac{\partial s}{\partial t} = -\sigma v'(s) \right)$

$u_t = -\sigma v'$

Also $u_x = v'(x - \sigma t)$ $\quad \left(u_x = \frac{dv}{ds} \frac{\partial s}{\partial x} = v'(s) \right)$

(1) —— $u_x = v'$

$u_{xx} = v''$

$u_{xxx} = v'''$

Since u is solution of eq^n (1), put these values in eq^n (1).

$$\therefore -\sigma v' + 6vv' + v''' = 0$$

↳ Here v ka derivative saath me hi h so we get $\frac{v^2}{2}$

Integrate this eq^n, we get

$$-\sigma v + \frac{6v^2}{2} + v'' = a$$

$$-\sigma v + 3v^2 + v'' = a \quad \text{—— (3)}$$

where a is constant of integration

Multiply (3) with v' we get

→ Here v' ka derivative v'' saath me h

$$-\sigma vv' + 3v^2 v' + v'v'' = av'$$

↳ Here v ka derivative v' saath me h

Integrate this eq^n, we get

$$-\frac{\sigma v^2}{2} + \frac{3v^3}{3} + \frac{(v')^2}{2} = av + b$$

$$-\frac{\sigma v^2}{2} + v^3 + \frac{(v')^2}{2} = av + b \quad \text{—— (4)}$$

where b is constant of integration

using energy methods which gives $u, u_x, u_{xx}, u_t \to 0$ as $x \to \pm\infty$

Now we solve eqn ① with the help of eqn ③ & ④ in such a way that v satisfy $v, v', v'' \longrightarrow 0$ as $s \to \pm\infty$ where $s = x - \sigma t$ — ⑤

Under what condn solitons used (in ⑤) → In compulsory Ques

Under this condition, eqn ③ & ④ gives us

$a = 0, b = 0$ as $s \to \pm\infty$

$\left\{ \begin{array}{l} s \text{ is fun}^n \text{ of } x-\sigma t \\ u \text{ is fun}^n \text{ of } x-\sigma t \\ \text{so } v \text{ is fun}^n \text{ of } s \end{array} \right\}$ → Jaise Jaise s ko $\pm\infty$ tak krege then v, v', v'' ki value zero ho jaygi

$$\left[\begin{array}{l} u = u(x,t) = v(x-\sigma t) \\ \qquad = v(s) \\ \text{is called solitary wave solution as } s \to \pm\infty \end{array} \right]$$

Now, eqn ③ and ④ becomes

$$-\sigma v + 3v^2 + v'' = 0 \quad \text{— ⑤}$$

$$-\frac{\sigma v^2}{2} + v^3 + \frac{(v')^2}{2} = 0 \quad \text{— ⑥}$$

Now eqn ⑥ $\Rightarrow$ $-\frac{\sigma v^2}{2} + v^3 + \frac{(v')^2}{2} = 0$

$$\Rightarrow \frac{(v')^2}{2} = \frac{\sigma v^2}{2} - v^3$$

$$\frac{(v')^2}{2} = v^2\left(\frac{\sigma}{2} - v\right)$$

$$(v')^2 = v^2(\sigma - 2v)$$

$$\Rightarrow v' = \pm v\sqrt{\sigma - 2v} \quad \text{— (*)}$$

for convinience Take negative sign, we get

$$v' = -v\sqrt{\sigma - 2v}$$

$$\Rightarrow \frac{dv}{ds} = -v\sqrt{\sigma - 2v}$$

$$\Rightarrow \frac{-dv}{v\sqrt{\sigma - 2v}} = ds$$

Integrating both sides, we have

$$\int \frac{-dv}{v\sqrt{\sigma - 2v}} + c = \int ds$$

Put $v = \frac{\sigma}{2}\operatorname{sech}^2\theta$

$$\Rightarrow dv = +\frac{\sigma}{2} \cdot 2\operatorname{sech}\theta(-\operatorname{sech}\theta \tanh\theta)\,d\theta$$

$\because \frac{d}{d\theta}(\operatorname{sech}\theta) = -\operatorname{sech}\theta\tanh\theta$

$$-dv = \sigma\operatorname{sech}^2\theta\tanh\theta\,d\theta$$

Also, $v\sqrt{\sigma - 2v} = \frac{\sigma}{2}\operatorname{sech}^2\theta \sqrt{\sigma - \sigma\operatorname{sech}^2\theta}$

$= \frac{\sigma^{3/2}}{2}\operatorname{sech}^2\theta \tanh\theta$

In simple $\sec^2\theta - \tan^2\theta = 1$ but in hyperbolic $[\because \operatorname{sech}^2\theta + \tanh^2\theta = 1]$ & $\cosh^2\theta - \sinh^2\theta = 1$

Put these value in eqn (7), we get

$$\int \frac{2}{\sigma^{1/2}}\,d\theta + c = s$$

$$\Rightarrow \frac{2}{\sqrt{\sigma}}\theta + C = s \quad \text{—— (8)}$$

Now, $v(s) = \frac{\sigma}{2}\operatorname{sech}^2\theta$

from eqn (8), $\theta = \frac{\sqrt{\sigma}(s-c)}{2}$

Then above eqn become

$$v(s) = \frac{\sigma}{2}\operatorname{sech}^2\left(\frac{\sqrt{\sigma}(s-c)}{2}\right) \quad \text{—— (9)}$$

Hence the solⁿ of eqn (1) is given by

$u = u(x,t)$

$= v(x - \sigma t)$ $= \frac{\sigma}{2}\left[\operatorname{sech}\left(\frac{\sqrt{\sigma}(x-\sigma t - c)}{2}\right)\right]^2$

$= v(s)$ $= D\operatorname{sech}[E(x - \sigma t - C)]^2$

$= \frac{\sigma}{2}\operatorname{sech}^2\left(\frac{(s-c)\sqrt{\sigma}}{2}\right)$ $= D\operatorname{sech}[E(s-C)]^2$ for some constant c and $s = x - \sigma t$

This solution is called Solitones.

Remark:- Similarly, by taking positive sign in eqn (7) we get Solitone in cosh form.

Conversion of non linear PDE into linear PDE

imp

OR

Cole-Hopf Transformation

Its topic se aur iske baad vale jo bi article h unme se 8 Marks ka ques aayga

Let us consider a quasi linear parabolic equation

$u_t - a\Delta u = -b|Du|^2 \rightarrow$ $u_t - a\Delta u + b|Du|^2 = 0$ in $\mathbb{R}^n \times (0,\infty)$ $\quad a > 0$

$u = g$ on $\mathbb{R}^n \times \{t = 0\}$ $\quad$ —— (1)

↳ so it is non linear

where a, b are constants and g is a known function

Let us consider $u = u(x,t)$ be solution of eqn (1)

To convert eqn (1) into linear PDE, we get

[If a fun^n is analytic in space

$w = \phi(u)$ where $\phi : \mathbb{R} \to \mathbb{R}$ is a smooth function which is
Jine u ko w ke saath connect
be determine such that w satisfy linear PDE

[How to find the value of ϕ]

Now, $w = \phi(u)$

Differentiate w.r.t 't'

$$w_t = \phi'(u) u_t$$

Also $\Delta w = \sum_{i=1}^{n} \frac{\partial^2 w}{\partial x_i^2}$, ——— (*)

$$\Delta = \nabla^2 = D^2 = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$$

Now, $\frac{\partial w}{\partial x_i} = \phi'(u) \frac{\partial u}{\partial x_i} \quad \forall i$

$$\Rightarrow \frac{\partial^2 w}{\partial x_i^2} = \phi''(u) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_i} + \phi'(u) \frac{\partial^2 u}{\partial x_i^2}$$

$$= \phi''(u) \left(\frac{\partial u}{\partial x_i} \right)^2 + \phi'(u) \frac{\partial^2 u}{\partial x_i^2}$$

Taking summation on both side

$$\sum_{i=1}^{n} \frac{\partial^2 w}{\partial x_i^2} = \sum_{i=1}^{n} \left[\phi''(u) \left(\frac{\partial u}{\partial x_i} \right)^2 + \phi'(u) \frac{\partial^2 u}{\partial x_i^2} \right]$$

$$\Delta w = \phi''(u) \sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i} \right)^2 + \phi'(u) \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$$

$$\Delta w = \phi''(u) |Du|^2 + \phi'(u) \Delta u \quad \text{——— (2)}$$

$$\left[x = (x_1, x_2, \dots, x_n) \quad \sum_{i=1}^{n} x_i^2 = |x|^2 \right]$$

Consider, (1)

$$w_t = \phi'(u) u_t$$

$$= \phi'(u) (a\Delta u - b|Du|^2) \quad \text{(By eq}^n \text{ (1))}$$

$$= a\phi'(u)\Delta u - b\phi'(u)|Du|^2$$

$$= a[\Delta w - \phi''(u)|Du|^2] - b\phi'(u)|Du|^2 \quad \text{(from eq}^n \text{ (2))}$$

$$= a\Delta w - a\phi''(u)|Du|^2 - b\phi'(u)|Du|^2$$

$$w_t = a\Delta w - |Du|^2 (a\phi''(u) + b\phi'(u))$$

$$w_t - a\Delta w = -|Du|^2 (a\phi''(u) + b\phi'(u))$$

Now, w satisfy linear (equation)

$$w_t - a\Delta w = 0 \Leftrightarrow -|Du|^2 (a\phi''(u) + b\phi'(u)) = 0$$

$w_t = a\Delta w$ providing $a\phi''(u) + b\phi'(u) = 0$

Now u satisfies the eq^n

$$a\phi''(u) + b\phi'(u) = 0$$

Now we try to find the sol^n this equation

$\Leftrightarrow \quad a\phi''(u) + b\phi'(u) = 0$

$\Leftrightarrow \quad a\phi''(u) = -b\phi'(u)$

$\Leftrightarrow \quad \dfrac{\phi''(u)}{\phi'(u)} = \dfrac{-b}{a}$

Integration both side

$\Leftrightarrow \quad \log \phi'(u) = \dfrac{-b}{a} u + \log c$

$\Leftrightarrow \quad \log \phi'(u) - \log c = -\dfrac{b}{a} u$

$\Leftrightarrow \quad \log \dfrac{\phi'(u)}{c} = -\dfrac{b}{a} u$

$\Leftrightarrow \quad \dfrac{\phi'(u)}{c} = e^{\frac{-bu}{a}}$

$\Leftrightarrow \quad \phi'(u) = c e^{\frac{-bu}{a}}$ Integration both side

$\Leftrightarrow \quad \phi(u) = \dfrac{c e^{\frac{-bu}{a}}}{-b/a} + d$

$\Leftrightarrow \quad \phi(u) = -\dfrac{ac}{b} e^{\frac{-bu}{a}} + d$

↳ Isse yha 1 bnana h so we put $c = \frac{-b}{a}$

In particular, set $d=0$ and $c = \dfrac{-b}{a}$

then w satisfy $w_t - a\Delta w = 0 \Leftrightarrow \phi(u) = e^{\frac{-bu}{a}}$ ——— (3)

Hence w satisfy Initial boundary value problem

$$\left.\begin{aligned} & w_t - a\Delta w = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty) \\ & w = e^{\frac{-bg}{a}} \quad \text{on } \mathbb{R}^n \times \{t=0\} \end{aligned}\right\} \text{ ——— (4)}$$

$w = \phi(u) = e^{\frac{-bu}{a}}$

In eqn (1), on boundary $u = g$

$\therefore$ we get $w = e^{\frac{-bg}{a}}$

[By eqn (1) $u = g$ on $\mathbb{R}^n \times \{t=0\}$ and $w = \phi(u) = e^{\frac{-bu}{a}}$ on $\mathbb{R}^n \times \{t=0\}$]

The solution of eqn (4) is given by

$$w(x,t) = \frac{1}{(4\pi a t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4at}} e^{\frac{-bg(y)}{a}}\, dy \quad \text{——— (5)}$$

[By fundamental soln of heat eqn] IVP

Now solution of eqn (1) is given by

$$u = -\frac{a}{b} \log w$$

$w = \phi(u) = e^{\frac{-bu}{a}}$

$\log w = \dfrac{-bu}{a}$

$u = -\dfrac{a}{b} \log w$

i.e. $u(x,t) = -\frac{a}{b} \log \left[\frac{1}{(4\pi a t)^{n/2}} \int_{\mathbb{R}^n} e^{\frac{-|x-y|^2}{4at}} e^{\frac{-bg}{a}} dy \right]$

is the required solution of eqn (1) and the transformation $w = \phi(u) = e^{\frac{-bu}{a}}$ is called Cole-Hopf transformation.

Potential function:

Potential function is used to convert a non linear PDE system into a single linear PDE

Let us consider an example of inviscid, incompressible fluid flow

$$\left.\begin{array}{ll} \vec{u}_t + \vec{u}\cdot D\vec{u} = -Dp + \vec{f} & \text{in } \mathbb{R}^3 \times (0,\infty) \\ \operatorname{div}\vec{u} = 0 & \text{in } \mathbb{R}^3 \times (0,\infty) \\ \vec{u} = \vec{g} & \text{on } \mathbb{R}^3 \times \{t=0\} \end{array}\right\} \quad \text{--- (1)}$$

Here the velocity field $u = (u_1, u_2, u_3)$ is unknown function and scalar pressure p, external force $f = (f_1, f_2, f_3)$ and the initial velocity $g = (g_1, g_2, g_3)$ are known function, D denotes gradiant of u w.r.t. $x = (x_1, x_2, x_3)$

Now we convert system (1) of non linear PDE into a single linear PDE as follow:

Let us consider $\operatorname{div} g = 0$ ← Learn

and $f = Dh$ for some scalar function $h : \mathbb{R}^3 \times (0,\infty) \to \mathbb{R}$ } (2)

$f = Dh$ means that the external force f is derived from potential.

We try to find the solution of eqn (1) u in the form of

$\underbrace{u = D v}_{\text{i.e. } u \text{ is derive from } v}$ —— (3)

Here function v is called potential function.

Also $\nabla \times u = D \times u$

$= D \times (D v)$ (from eqn (3))

$= 0$

$$\begin{vmatrix} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ v_1 & v_2 & v_3 \end{vmatrix} = 0$$

⇒ u is a irrotational function

and eqn (1) gives $\operatorname{div} u = 0$

⇒ $D \cdot u = 0$

$\Rightarrow D.(D.v) = 0$

$\Rightarrow D^2 v = 0$

$\Rightarrow \Delta v = 0$

$\Rightarrow v$ is a harmonic function

Also from eqn (1)

$u = g$ on $\mathbb{R}^3 \times \{t=0\}$

$\Rightarrow u(x,0) = g(x)$ $\quad$ [$u = u(x,t)$ At $t=0$ then $u(x,0)$]

$\Rightarrow D.v(x,0) = g(x)$

i.e, $D.v(\cdot,0) = g(\cdot)$

and from eqn (1)

$u_t + u\,Du = -Dp + f$

$\Rightarrow D.v_t + D|D.v|^2 = -Dp + Dh$

$\Rightarrow D.(v_t + |D.v|^2) = D(-p+h)$

$\Rightarrow v_t + |D.v|^2 = -p + h$ —— (4)

$u = D.v$
$u_t = D.v_t$
Also $u.Du$
$\Rightarrow (D.v).(DD.v)$
$= D(D.v)(D(v))$
$= D|D.v|^2$

This eqn is called Bernoulli's law

Since the function p and h are known function

$\therefore$ We can easily find the value of v from eqn (4)

Eqn (4) is a linear partial differential eqn providing v is harmonic and $D.v = g$ on $\mathbb{R}^3 \times \{t=0\}$

2013
Hodograph Transformation

Hodograph Transformation is a technique to convert a quassi linear system of PDE into linear system by interchanging the role of dependent and independent variables.

Consider a two dimensional irrotational fluid flow

$$(\sigma^2(\vec{u}) - \underbrace{(u^1)^2}_{\text{dependent}})\, u^1_{x_1} - u^1 u^2 (u^1_{x_2} + u^2_{x_1}) + (\sigma^2(\vec{u}) - (u^2)^2)\, u^2_{x_2} = 0$$

$$u^1_{x_2} - u^2_{x_1} = 0$$
—(1) in $\mathbb{R}^2$

Here the velocity field $\vec{u} = (u^1, u^2)$ is unknown and σ denote the speed of the wave, superscript variable are dependent and subscript variable are independent.

To apply Hodograph Transformation we convert the role of dependent variables into independent variables and vice-versa.

In eqn ①

$$\left.\begin{aligned} u^1 &= u^1(x_1, x_2) \\ u^2 &= u^2(x_1, x_2) \end{aligned}\right\} \text{ ②}$$

represents u^1, u^2 are depending on x_1 and x_2.

After interchanging there role, we get [dependent convert into independent & vice versa]

$$\left.\begin{aligned} x^1 &= x^1(u_1, u_2) \\ x^2 &= x^2(u_1, u_2) \end{aligned}\right\} \text{ — ③}$$

providing $J = \dfrac{\partial(u^1, u^2)}{\partial(x_1, x_2)} \neq 0$ in $\mathbb{R}^2$ —— ④

$$\frac{\partial(u^1,u^2)}{\partial(x_1,x_2)} = \begin{vmatrix} \frac{\partial u^1}{\partial x_1} & \frac{\partial u^1}{\partial x_2} \\ \frac{\partial u^2}{\partial x_1} & \frac{\partial u^2}{\partial x_2} \end{vmatrix} = \begin{vmatrix} u^1_{x_1} & u^1_{x_2} \\ u^2_{x_1} & u^2_{x_2} \end{vmatrix}$$

i.e, $u^1_{x_1} u^2_{x_2} - u^1_{x_2} u^2_{x_1} \neq 0$

Now, $J . x^1_{u_1} = (u^1_{x_1} u^2_{x_2} - u^1_{x_2} u^2_{x_1}) . x^1_{u_1}$

$$= \left(\frac{\partial u^1}{\partial x_1}\frac{\partial u^2}{\partial x_2} - \frac{\partial u^1}{\partial x_2}\frac{\partial u^2}{\partial x_1}\right)\frac{\partial x^1}{\partial u_1}$$

$$= \frac{\partial u^1}{\partial x_1}\frac{\partial u^2}{\partial x_2}\frac{\partial x^1}{\partial u_1} - \frac{\partial u^1}{\partial x_2}\frac{\partial u^2}{\partial x_1}\frac{\partial x^1}{\partial u_1}$$

{ $u = r\cos\theta$ only variable change $u^1 = r\cos\theta_1$ }

Only same cancel out & cancel two terms together

$$= \frac{\partial u^2}{\partial x_2} - \frac{\partial u^1}{\partial x_2}\frac{\partial u^2}{\partial u_1}$$

[By chain rule]

Agr yha ∂u_2 vali term cancel krte too x_1 & x_2 are independent hota

$$= \frac{\partial u^2}{\partial x_2} - 0$$

(b) [$\because u^1$ and u^2 are independent of each other]

$= u^2_{x_2}$ ⓑ ~~and~~ ⑤(a)

$J . x^2_{u_2} = (u^1_{x_1} u^2_{x_2} - u^1_{x_2} u^2_{x_1}) . x^2_{u_2}$

$$= \left(\frac{\partial u^1}{\partial x_1}\frac{\partial u^2}{\partial x_2} - \frac{\partial u^1}{\partial x_2}\frac{\partial u^2}{\partial x_1}\right)\frac{\partial x^2}{\partial u_2}$$

$$= \frac{\partial u^1}{\partial x_1}\frac{\partial u^2}{\partial x_2}\frac{\partial x^2}{\partial u_2} - \frac{\partial u^1}{\partial x_2}\frac{\partial u^2}{\partial x_1}\frac{\partial x^2}{\partial u_2}$$

$$= \frac{\partial u^1}{\partial x_1} - \frac{\partial u^1}{\partial u_2}\frac{\partial u^2}{\partial x_1}$$

$$= \frac{\partial u^1}{\partial x_1} - 0$$

[$\because u^1$ and u^2 are independent of each other]

$$= \frac{\partial u^1}{\partial x_1}$$

$$= u^1_{x_1} \quad \text{—— (b)}$$

$$J \cdot x^1_{u_2} = (u^1_{x_1} u^2_{x_2} - u^1_{x_2} u^2_{x_1}) \cdot x^1_{u_2}$$

$$= \left(\frac{\partial u^1}{\partial x_1} \frac{\partial u^2}{\partial x_2} - \frac{\partial u^1}{\partial x_2} \frac{\partial u^2}{\partial x_1} \right) \frac{\partial x^1}{\partial u_2}$$

$$= \frac{\partial u^1}{\cancel{\partial x_1}} \frac{\partial u^2}{\partial x_2} \frac{\cancel{\partial x^1}}{\partial u_2} - \frac{\partial u^1}{\partial x_2} \frac{\partial u^2}{\cancel{\partial x_1}} \frac{\cancel{\partial x^1}}{\cancel{\partial u_2}}$$

$$= \frac{\partial u^1}{\partial u_2} \frac{\partial u^2}{\partial x_2} - \frac{\partial u^1}{\partial x_2}$$

$$= 0 - \frac{\partial u^1}{\partial x_2} \qquad [\because u^1 \text{ and } u^2 \text{ are independent of each other}]$$

$$= -\frac{\partial u^1}{\partial x_2}$$

$$= -u^1_{x_2} \quad \text{—— (c)}$$

$$J \cdot x^2_{u_1} = (u^1_{x_1} u^2_{x_2} - u^1_{x_2} u^2_{x_1}) \cdot x^2_{u_1}$$

$$= \left(\frac{\partial u^1}{\partial x_1} \frac{\partial u^2}{\partial x_2} - \frac{\partial u^1}{\partial x_2} \frac{\partial u^2}{\partial x_1} \right) \frac{\partial x^2}{\partial u_1}$$

$$= \frac{\partial u^1}{\partial x_1} \frac{\partial u^2}{\cancel{\partial x_2}} \frac{\cancel{\partial x^2}}{\partial u_1} - \frac{\cancel{\partial u^1}}{\cancel{\partial x_2}} \frac{\partial u^2}{\partial x_1} \frac{\cancel{\partial x^2}}{\cancel{\partial u_1}}$$

$$= \frac{\partial u^1}{\partial x_1} \frac{\partial u^2}{\partial u_1} - \frac{\partial u^2}{\partial x_1}$$

$$= 0 - \frac{\partial u^2}{\partial x_1} \qquad [\because u^1 \text{ and } u^2 \text{ are independent of each other}]$$

$$= -\frac{\partial u^2}{\partial x_1}$$

$$= -u^2_{x_1} \quad \text{—— (d)}$$

Put the value of eqn (a), (b), (c) and (d) in eqn (1)

$$(\sigma^2(\vec{u}) - \underbrace{(u_1)^2}_{\text{independent}}) J \cdot x^2_{u_2} - u_1 u_2 (-J x^1_{u_2} - J x^2_{u_1}) + (\sigma^2(\vec{u}) - (u_2)^2) J x^1_{u_1} = 0$$

and $-J \cdot x^1_{u_2} - (-J x^2_{u_1}) = 0$

Cancelling J from above eqn, we get

$$\left. \begin{aligned} &(\sigma^2(\vec{u}) - (u_1)^2) x^2_{u_2} + u_1 u_2 (x^1_{u_2} + x^2_{u_1}) + (\sigma^2(\vec{u}) - (u_2)^2) x^1_{u_1} = 0 \\ &\text{and } \quad x^1_{u_2} - x^2_{u_1} = 0 \end{aligned} \right\} \text{—(3)} \rightarrow \text{linear}$$

[which is a linear system of PDE in variable $x = (x^1, x^2)$]

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as a function of a variable $u = (u_1, u_2)$.

Legendre Transformation :-

"Legendre Transformation is a technique to convert quassi linear system of PDE into linear system by interchanging the role of dependent and independent variables.

Consider a minimal surface equation

$$(1+(u_{x_2})^2)u_{x_1x_1} - 2u_{x_1}u_{x_2}u_{x_1x_2} + (1+(u_{x_1})^2)u_{x_2x_2} = 0) \quad \text{——} \quad (1)$$

↳ we find these three valu

Here the function $u = (u^1, u^2)$ is unknown

and define $p = Du \quad \Rightarrow (p^1, p^2) = (Du_{x_1}, Du_{x_2})$

i.e, $p^1 = u_{x_1}$

$p^2 = u_{x_2}$

Now we interchange the role of dependent variables p^1, p^2

$$\left.\begin{aligned} \text{i.e, } p^1 &= u_{x_1}(x_1, x_2) \\ p^2 &= u_{x_2}(x_1, x_2) \end{aligned}\right\} \quad \text{——} \quad (2)$$

and independent variables x^1, x^2

$$\left.\begin{aligned} x^1 &= x^1(p_1, p_2) \\ x^2 &= x^2(p_1, p_2) \end{aligned}\right\} \quad \text{——} \quad (3)$$

provided $J = \dfrac{\partial(p^1, p^2)}{\partial(x_1, x_2)} \neq 0$

$$\begin{vmatrix} \dfrac{\partial p^1}{\partial x_1} & \dfrac{\partial p^1}{\partial x_2} \\ \dfrac{\partial p^2}{\partial x_1} & \dfrac{\partial p^2}{\partial x_2} \end{vmatrix}$$

Put $p^1 = \dfrac{\partial u}{\partial x_1}$ & $p^2 = \dfrac{\partial u}{\partial x_2}$

i.e, $u^1_{x_1x_1} u^2_{x_2x_2} - u^1_{x_1x_2} u^2_{x_1x_2} \neq 0$

↳ is funⁿ of p_1, p_2

Define $v(p) = x(p).p - u(x(p))$ —— (4)

↳ is funⁿ of x_1, x_2

$= (x^1, x^2).(p_1, p_2) - u(x(p))$

↳ u is funⁿ of x_1 & x_2

$= x^1 p_1 + x^2 p_2 - u(x(p))$

First we claim that

$J . v_{p_2p_2} = u_{x_1x_1}$

Consider $v_{p_2} = \dfrac{\partial x^1}{\partial p_2} p_1 + x^1 \dfrac{\partial p_1}{\partial p_2} + \dfrac{\partial x^2}{\partial p_2} p_2 + x^2 \dfrac{\partial p_2}{\partial p_2} - \left(\dfrac{\partial u}{\partial x^1}\dfrac{\partial x^1}{\partial p_2} + \dfrac{\partial u}{\partial x^2}\dfrac{\partial x^2}{\partial p_2}\right)$

(p_1 & p_2 independent) ⇓ 1

$= \dfrac{\partial x^1}{\partial p_2} p_1 + x^2 + \dfrac{\partial x^2}{\partial p_2} p_2 - p_1 \dfrac{\partial x^1}{\partial p_2} - p_2 \dfrac{\partial x^2}{\partial p_2}$

$= x^2$

[∵ p_1 & p_2 are independent
∴ $\dfrac{\partial p_1}{\partial p_2} = 0$]

Differentiate again w.r.t. p_2, we have

$$v_{p_2p_2} = \frac{\partial x^2}{\partial p_2}$$

Multiply Jacobian J on both side

$$J \cdot v_{p_2p_2} = J \cdot \frac{\partial x^2}{\partial p_2}$$

$$= \left(u_{x_1x_1} u_{x_2x_2} - u_{x_1x_2} u_{x_1x_2}\right) \frac{\partial x^2}{\partial p_2}$$

$$= \left(\frac{\partial}{\partial x_1}\left(\frac{\partial u}{\partial x_1}\right) \frac{\partial}{\partial x_2}\left(\frac{\partial u}{\partial x_2}\right) - \frac{\partial}{\partial x_1}\left(\frac{\partial u}{\partial x_2}\right) \frac{\partial}{\partial x_2}\left(\frac{\partial u}{\partial x_1}\right)\right) \frac{\partial x^2}{\partial p_2}$$

$$= \left(\frac{\partial p^1}{\partial x_1} \frac{\partial p^2}{\partial x_2} - \frac{\partial p^2}{\partial x_1} \frac{\partial p^1}{\partial x_2}\right) \frac{\partial x^2}{\partial p_2}$$

$$= \frac{\partial p^1}{\partial x_1} \cancel{\frac{\partial p^2}{\partial x_2}} \cancel{\frac{\partial x^2}{\partial p_2}} - \frac{\partial p^2}{\partial x_1} \cancel{\frac{\partial p^1}{\partial x_2}} \cancel{\frac{\partial x^2}{\partial p_2}}$$

$$= \frac{\partial p^1}{\partial x_1} - \frac{\partial p^2}{\partial x_1} \frac{\partial p^1}{\partial p_2}$$

$$= \frac{\partial p^1}{\partial x_1} - 0$$ [$\because p^1$ and p^2 are independent of each other]

$$= \frac{\partial p^1}{\partial x_1}$$

$$= \frac{\partial}{\partial x_1}\left(\frac{\partial u}{\partial x_1}\right)$$

$$= \frac{\partial^2 u}{\partial x_1^2}$$

$$= u_{x_1x_1}$$

$$J \cdot v_{p_2p_2} = u_{x_1x_1} \quad \text{—— (A)}$$

Similarly, $J \cdot v_{p_1p_1} = u_{x_2x_2}$ —— (B)

Now we claim that $J \cdot v_{p_1p_2} = -u_{x_1x_2}$ ← In 8 Marks, we write direct

Consider $v_{p_1} = \frac{\partial x^1}{\partial p_1} p_1 + x^1 \frac{\partial p_1}{\partial p_1} + x^2 \frac{\partial p_2}{\partial p_1} + \frac{\partial x^2}{\partial p_1} p_2 - \left(\underbrace{\frac{\partial u}{\partial x^1}}_{p^1} \frac{\partial x^1}{\partial p_1} + \underbrace{\frac{\partial u}{\partial x^2}}_{p^2} \frac{\partial x^2}{\partial p_1}\right)$

$$= \cancel{\frac{\partial x^1}{\partial p_1} p_1} + x^1 + \cancel{p_2 \frac{\partial x^2}{\partial p_1}} - \cancel{p_1 \frac{\partial x^1}{\partial p_1}} - \cancel{p_2 \frac{\partial x^2}{\partial p_1}}$$

$v_{p_1} = x^1$

[$\because p_1$ & p_2 are independent $\therefore \frac{\partial p_2}{\partial p_1} = 0$]

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Differentiate again w.r.t. p_2, we get

$$v_{p_1p_2} = \frac{\partial x^1}{\partial p_2}$$

Multiply Jacobian J on both side

$$J v_{p_1p_2} = J.\frac{\partial x^1}{\partial p_2}$$

$$= (u_{x_1x_1}u_{x_2x_2} - u_{x_1x_2}u_{x_1x_2})\frac{\partial x^1}{\partial p_2}$$

$$= \left[\frac{\partial}{\partial x_1}\left(\frac{\partial u}{\partial x_1}\right)\frac{\partial}{\partial x_2}\left(\frac{\partial u}{\partial x_2}\right) - \frac{\partial}{\partial x_1}\left(\frac{\partial u}{\partial x_2}\right)\frac{\partial}{\partial x_2}\left(\frac{\partial u}{\partial x_1}\right)\right]\frac{\partial x^1}{\partial p_2}$$

$$= \frac{\partial p^1}{\cancel{\partial x_1}}\frac{\partial p^2}{\partial x_2}\frac{\cancel{\partial x^1}}{\partial p_2} - \cancel{\frac{\partial p^2}{\partial x_1}}\frac{\partial p^1}{\partial x_2}\cancel{\frac{\partial x^1}{\partial p_2}}$$

$$= \frac{\partial p^1}{\partial p_2}\frac{\partial p^2}{\partial x_2} - \frac{\partial p^1}{\partial x_2}$$

$$= 0 - \frac{\partial p^1}{\partial x_2}$$ [∵ p^1 and p^2 are independent of each other]

$$= -\frac{\partial p^1}{\partial x_2}$$

$$= -\frac{\partial}{\partial x_2}\left(\frac{\partial u}{\partial x_1}\right)$$

$$= -\frac{\partial^2 u}{\partial x_2 \partial x_1}$$

$$= -\frac{\partial^2 u}{\partial x_1 \partial x_2}$$

$$J v_{p_1p_2} = -u_{x_1x_2} \quad \text{—— (C)}$$

Put the value of (A), (B), (C) in eqn (1), we get

$$(1+(p_2)^2)J.v_{p_2p_2} + 2p_1p_2 J.v_{p_1p_2} + (1+(p_1)^2)J.v_{p_1p_1} = 0$$

Cancelling J from above eqn we have

$$\Rightarrow (1+(p_2)^2)v_{p_2p_2} + 2p_1p_2 v_{p_1p_2} + (1+(p_1)^2)v_{p_1p_1} = 0$$

↳ Iske saath v ka koi derivative mult aur na hi koi ... so, it is linear.

which is the required linear P.D.E

Ques:- Using $\xi = x+t$ and $\eta = x-t$

Prove that $u_{tt} - u_{xx} = 0 \Leftrightarrow u_{\xi\eta} = 0$

Solⁿ:- It is given that $\xi = x+t$ and $\eta = x-t$

$$\Rightarrow x = \frac{\xi+\eta}{2}, \quad t = \frac{\xi-\eta}{2}$$

First we consider $u_{tt} - u_{xx} = 0$ and prove that $u_{\xi\eta} = 0$

Let us consider $u = u(x,t)$

Differentiate w.r.t. ξ, we have

$$u_\xi = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial \xi}$$

$$u_\xi = \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t}\right)$$

Differentiate w.r.t. η, we get

$$u_{\xi\eta} = \frac{1}{2}\left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t}\right)\frac{\partial x}{\partial \eta} + \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t}\right)\frac{\partial t}{\partial \eta}\right]$$

$$= \frac{1}{2}\left[\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial t}\right)\left(\frac{1}{2}\right) + \left(\frac{\partial^2 u}{\partial t \partial x} + \frac{\partial^2 u}{\partial t^2}\right)\left(\frac{-1}{2}\right)\right]$$

$$= \frac{1}{4}\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial t} - \frac{\partial^2 u}{\partial t \partial x} - \frac{\partial^2 u}{\partial t^2}\right]$$

$$= \frac{1}{4}\left[u_{xx} - u_{tt} + u_{xt} - u_{tx}\right]$$

$$= \frac{1}{4}\left[u_{xx} - u_{tt}\right]$$

[$\because$ Second order derivation of function u exist $\therefore u_{xt} = u_{tx}$]

$$= 0 \qquad [\because u_{tt} - u_{xx} = 0]$$

$\Rightarrow u_{\xi\eta} = 0$

Conversely, let us consider $u_{\xi\eta} = 0$

RTP: $u_{tt} - u_{xx} = 0$

Let us consider $u = u(\xi, \eta)$

Differentiate w.r.t. 't', we get

$$u_t = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial t}$$

$$= \frac{\partial u}{\partial \xi}\cdot 1 + \frac{\partial u}{\partial \eta}(-1)$$

$$= \frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta}$$

$$u_t = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial t} = \frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta}$$

$$u_{tt} = \frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta}\right)\frac{\partial \xi}{\partial t} + \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta}\right)\frac{\partial \eta}{\partial t}$$

Again diff. w.r.t. 't', we get

$$u_{tt} = \frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta}\right)\frac{\partial \xi}{\partial t} + \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta}\right)\frac{\partial \eta}{\partial t}$$

$$= \frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta}\right)\cdot 1 + \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta}\right)(-1)$$

$$= \frac{\partial^2 u}{\partial \xi^2} - \frac{\partial^2 u}{\partial \xi \partial \eta} - \frac{\partial^2 u}{\partial \eta \partial \xi} + \frac{\partial^2 u}{\partial \eta^2}$$

$$= \left(\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2}\right) - 2\frac{\partial^2 u}{\partial \xi \partial \eta}$$

$\because u = u(\xi, \eta)$

Diff. w.r.t. x, we have

$$u_x = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x}$$

$$= \frac{\partial u}{\partial \xi}\cdot 1 + \frac{\partial u}{\partial \eta}\cdot 1$$

$$u_x = \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}$$

Again diff. w.r.t. x, we get

$$u_{xx} = \frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right)\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right)\frac{\partial \eta}{\partial x}$$

$$= \frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right)\cdot 1 + \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right) 1$$

$$= \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta \partial \xi} + \frac{\partial^2 u}{\partial \eta^2}$$

$$u_{xx} = \left(\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2}\right) + 2\frac{\partial^2 u}{\partial \eta \partial \xi}$$

Now,

$$u_{tt} - u_{xx} = \left(\cancel{\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2}}\right) - 2\frac{\partial^2 u}{\partial \xi \partial \eta} - \left(\cancel{\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2}}\right) - 2\frac{\partial^2 u}{\partial \xi \partial \eta}$$

$$= -4u_{\xi\eta}$$

$$= -4(0) \quad [\because u_{\xi\eta} = 0 \text{ (Given)}$$

$$= 0$$

$\therefore u_{tt} - u_{xx} = 0$

Most Imp

Scaling Invariant Solution

OR

Similarity under scaling

OR

2016, 2015, 2013

Eqn ke name se aayga ye article

~~Consider~~

Find the solution of Parous medium eqn

Consider a Parous medium eqn

$$u_t - \Delta u^r = 0 \text{ in } \mathbb{R}^n \times (0, \infty) \quad (1)$$

where $r > 1$ is a constant and $u \geq 0$

We try to find out the solution of eqn (1) of the form

$$u(x,t) = \frac{1}{t^\alpha} v\left(\frac{x}{t^\beta}\right) \quad (2)$$

$x \in \mathbb{R}^n$, $t > 0$, α, β are constant

Let, $y = \frac{x}{t^\beta}$

$\Rightarrow u(x,t) = \frac{1}{t^\alpha} v(y)$ — ③ is the solution of eqn ①

↳ $\mathbb{R}^n$ dim. me h

Now, $u_t = \frac{-\alpha}{t^{\alpha+1}} v(y) + \frac{1}{t^\alpha} D v(y) \cdot \left(\frac{-\beta x}{t^{\beta+1}} \right)$ [By solⁿ of heat eqⁿ]

[$\mathbb{R}^n$ dim. me h too isk 1-D me toone ko fiye kisi normal vector se multiply krege. v is funⁿ of y &, $y = \frac{x}{t^\beta}$ so $\frac{\partial y}{\partial t} = \frac{-\beta x}{t^{\beta+1}}$]

$$= \frac{-\alpha v(y)}{t^{\alpha+1}} - \frac{\beta}{t^{\alpha+1}} D v(y) \cdot \frac{x}{t^\beta}$$

$$u_t = \frac{-\alpha v(y)}{t^{\alpha+1}} - \frac{\beta}{t^{\alpha+1}} D v(y) \cdot y \quad \text{— Ⓐ}$$

Now, $u^\gamma(x,t) = \frac{1}{t^{\alpha\gamma}} v^\gamma(y)$ [We find $\Delta u^\gamma = \sum_{i=1}^{n} \frac{\partial^2 u^\gamma}{\partial x_i^2}$]

Differentiate w.r.t. x_i, we have

$$\frac{\partial u^\gamma}{\partial x_i} = \frac{1}{t^{\alpha\gamma}} \frac{\partial v^\gamma(y)}{\partial y_i} \frac{\partial y_i}{\partial x_i}$$

$y = \frac{x}{t^\beta}$

$y_i = \frac{x_i}{t^\beta} \Rightarrow \frac{\partial y_i}{\partial x_i} = \frac{1}{t^\beta}$

$$= \frac{1}{t^{\alpha\gamma}} \frac{\partial v^\gamma(y)}{\partial y_i} \cdot \frac{1}{t^\beta}$$

$\because y_i = \frac{x_i}{t^\beta}$

$\therefore \frac{\partial y_i}{\partial x_i} = \frac{1}{t^\beta}$

$$\frac{\partial u^\gamma}{\partial x_i} = \frac{1}{t^{\alpha\gamma+\beta}} \frac{\partial v^\gamma(y)}{\partial y_i}$$

→ Agr. diff. krte h to β add ho jata h.

Differentiate again w.r.t. x_i, we get

$$\frac{\partial^2 u^\gamma}{\partial x_i^2} = \frac{1}{t^{\alpha\gamma+2\beta}} \frac{\partial^2 v^\gamma(y)}{\partial y_i^2}$$

Taking summation on both side

$$\sum_{i=1}^{n} \frac{\partial^2 u^\gamma}{\partial x_i^2} = \frac{1}{t^{\alpha\gamma+2\beta}} \sum_{i=1}^{n} \frac{\partial^2 v^\gamma(y)}{\partial y_i^2}$$

$$\Rightarrow \Delta u^\gamma = \frac{1}{t^{\alpha\gamma+2\beta}} \Delta v^\gamma \quad \text{— Ⓑ}$$

Put the value of eqn Ⓐ and Ⓑ in eqn ①, we get,

$$\frac{\alpha}{t^{\alpha+1}} v(y) - \frac{\beta}{t^{\alpha+1}} D v(y) \cdot y - \frac{1}{t^{\alpha r+2\beta}} \Delta v^{r}(y) = 0$$

$$\Rightarrow \frac{\alpha}{t^{\alpha+1}} v(y) + \frac{\beta}{t^{\alpha+1}} D v(y) \cdot y + \frac{1}{t^{\alpha r+2\beta}} \Delta v^{r}(y) = 0 \quad \text{——— (4)}$$

To reduce eq^n (4) into a simple expression involving only y

Put $\alpha + 1 = \alpha r + 2\beta$ ——— (*)

Then eq^n (4) becomes

$$\frac{\alpha}{t^{\alpha+1}} v(y) + \frac{\beta}{t^{\alpha+1}} D v(y) \cdot y + \frac{1}{t^{\alpha+1}} \Delta v^{r}(y) = 0$$

$$\Rightarrow \alpha v(y) + \beta \, D v(y) \cdot y + \Delta v^{r}(y) = 0 \quad \text{——— (5)}$$

↳ Now we find both these value

∴ Now we reduce v(y) into a radial function by putting

$v(y) = w(r)$ where $r = |y|$

Now, $D v = \frac{dw}{dr} \frac{dr}{dy}$

$= \frac{dw}{dr} \frac{y}{r}$ $\quad \left\{ \because r = |y|, \; \frac{dr}{dy} = \frac{y}{|y|} = \frac{y}{r} \right.$

$D v \cdot y = \frac{dw}{dr} \frac{y^2}{r}$

$= \frac{dw}{dr} \frac{|y|^2}{r}$

$= \frac{dw}{dr} \frac{r^2}{r}$

$D v \cdot y = r \frac{dw}{dr}$

$D v \cdot y = r w'(r)$

Also $\Delta v^{r}(y) = \Delta w^{r}(r)$ where $r = |y|$

↳ v is fun^n of y $= \sum_{i=1}^{n} \frac{\partial^2}{\partial y_i^2} (w^{r}(r))$

so isliye yha y tgaya h

$$= \sum_{i=1}^{n} \frac{\partial}{\partial y_i} \left[\frac{\partial (w^{r}(r))}{\partial r} \frac{\partial r}{\partial y_i} \right]$$

$$= \sum_{i=1}^{n} \frac{\partial}{\partial y_i} \left[\frac{\partial (w^{r}(r))}{\partial r} \frac{y_i}{r} \right]$$

↳ Here $w^{r}(r)$, y_i & r all are fun^n of y_i

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$$= \sum_{i=1}^{n}\left[\frac{\partial^2(w^\gamma(r))}{\partial r^2}\,\frac{\partial r}{\partial y_i}\,\frac{y_i}{r} + \frac{\partial(w^\gamma(r))}{\partial r}\,\frac{1}{r} + \frac{\partial(w^\gamma(r))}{\partial r}\,y_i\left(\frac{-1}{r^2}\right)\frac{\partial r}{\partial y_i}\right]$$

$$\frac{\partial r}{\partial y_i} = \frac{y_i}{r}$$

$$= \sum_{i=1}^{n}\left[\frac{\partial^2(w^\gamma(r))}{\partial r^2}\,\frac{y_i^2}{r^2} + \frac{1}{r}\,\frac{\partial(w^\gamma(r))}{\partial r} - \frac{1}{r^3}\,\frac{\partial(w^\gamma(r))}{\partial r}\,y_i^2\right]$$

$$= \frac{1}{r^2}\,\frac{\partial^2(w^\gamma(r))}{\partial r^2}\sum_{i=1}^{n} y_i^2 + \frac{1}{r}\,\frac{\partial(w^\gamma(r))}{\partial r}\sum_{i=1}^{n} 1 - \frac{1}{r^3}\,\frac{\partial(w^\gamma(r))}{\partial r}\sum_{i=1}^{n} y_i^2$$

$$= \frac{1}{r^2}\,\frac{\partial^2(w^\gamma(r))}{\partial r^2}\,r^2 + \frac{1}{r}\,\frac{\partial(w^\gamma(r))}{\partial r}\cdot n - \frac{1}{r^3}\,\frac{\partial(w^\gamma(r))}{\partial r}\,r^2$$

$$= \frac{\partial^2 w^\gamma}{\partial r^2} + \frac{n}{r}\,\frac{\partial w^\gamma}{\partial r} - \frac{1}{r}\,\frac{\partial w^\gamma}{\partial r}$$

$$= \frac{\partial^2 w^\gamma}{\partial r^2} + \frac{(n-1)}{r}\,\frac{\partial w^\gamma}{\partial r}$$

$$\Delta v^\gamma(y) = (w^\gamma)'' + \frac{(n-1)}{r}(w^\gamma)'$$

Put both these value in eqn (5), we have,

$$\alpha w + \beta r w' + (w^\gamma)'' + \frac{(n-1)}{r}(w^\gamma)' = 0 \quad \text{— (6)}$$, where ' denotes $\frac{d}{dr}$

Set $\alpha = n\beta$ — (**)

∴ Eqn (6) becomes

$$n\beta w + \beta r w' + (w^\gamma)'' + \frac{(n-1)}{r}(w^\gamma)' = 0$$

Multiply by r^{n-1} both side, we have

$$\underbrace{n\beta r^{n-1} w + \beta r^n w'} + \underbrace{r^{n-1}(w^\gamma)'' + (n-1)r^{n-2}(w^\gamma)'} = 0$$

$$\beta(w r^n)' + (r^{n-1}(w^\gamma)')' = 0$$

Integrating on both side w.r.t 'r', we have

$$\beta w r^n + r^{n-1}(w^\gamma)' = a$$

By using radiation conditions i.e, $w, w' \to 0$ as $r \to \infty$

$\Rightarrow a = 0$

Hence above eqn becomes

$$\beta w r^n + r^{n-1}(w^\gamma)' = 0$$

$$r^{n-1}[\beta r w + (w^\gamma)'] = 0$$

$$\Rightarrow \beta r w + (w^\gamma)' = 0$$

$\Rightarrow (\omega^{r})' = -\beta \eta \omega$

$\Rightarrow r\omega^{r-1}\omega' = -\beta \eta \omega$

$\Rightarrow r\omega^{r-2}\omega' = -\beta \eta$

$\omega^{r-2}\omega' = -\dfrac{\beta \eta}{r}$

Integrating w.r.t. 'η', we have

$$\frac{\omega^{r-1}}{r-1} = \frac{-\beta}{r}\frac{\eta^2}{2} + b$$

$$\omega^{r-1} = b(r-1) - \frac{(r-1)\beta}{2r}\eta^2$$

$$\omega = \left[C - \frac{(r-1)\beta \eta^2}{2r}\right]^{\frac{1}{r-1}} \quad \text{where } C = b(r-1)$$

Hence solution of eq^n is given by

$$u(x,t) = \frac{1}{t^\alpha} v\left(\frac{x}{t^\beta}\right)$$

$$= \frac{1}{t^\alpha} v(y)$$

$$= \frac{1}{t^\alpha} \omega(\eta)$$

$$= \frac{1}{t^\alpha} \omega(|y|)$$

$$= \frac{1}{t^\alpha}\left[C - \frac{(r-1)\beta |y|^2}{2r}\right]^{\frac{1}{r-1}}$$

$$= \frac{1}{t^\alpha}\left[C - \frac{(r-1)\beta}{2r}\frac{|x|^2}{t^{2\beta}}\right]^{\frac{1}{r-1}} \quad \left[\text{where } |y| = \frac{|x|}{t^\beta}\right] \quad \text{—(A)}$$

is the required solution providing $\alpha + 1 = \alpha r + 2\beta$ and $\alpha = n\beta$

(by eq^n (*) and (**))

$\Rightarrow \alpha - \alpha r = 2\beta - 1$ and $\alpha = n\beta$

$\Rightarrow \alpha(1-r) = 2\beta - 1$ and $\alpha = n\beta$

$\Rightarrow n\beta(1-r) = 2\beta - 1$

$\Rightarrow n\beta(1-r) - 2\beta = -1$

$\Rightarrow \beta[2 - n(1-r)] = 1$

$\Rightarrow \beta = \dfrac{1}{2-n(1-\gamma)}$

$\Rightarrow \boxed{\beta = \dfrac{1}{2+n(\gamma-1)}} \rightarrow (B)$

The formulas (A), (B) & (C) are the Barenblatt–Kompaneetz–Zeldovich solution of porous medium equation.

Then $\alpha = \dfrac{n}{n(\gamma-1)+2} \rightarrow (C)$

Ques:- Find the solⁿ of $u_t - \Delta u^{5/3} = 0$

Solⁿ:- First we prove above article then comparing the value $u_t - \Delta u^{\gamma} = 0$ then we get γ

and given spac $u_t - \Delta u^{5/3} = 0$ in $\mathbb{R}^3 \times (0,\infty)$ then $n = 3$ $\gamma = \frac{5}{3}$

Then we find $\beta = \dfrac{1}{2+3\left(\frac{5}{3}-1\right)}$

In 16 Marks, we prove above article but in 8 marks we use direct result

Then put the value of β in (A) we get solⁿ.

Fourier Transformation in L^1 space:-

Defⁿ:- If $u \in L^1(\mathbb{R}^n)$, then we define fourier transformation of $u(x)$ by

$$\hat{u}(y) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix.y}\, u(x)\,dx \;;\; y \in \mathbb{R}^n$$

and its inverse fourier transformation is given by

$$\check{u}(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix.y}\, u(y)\,dy$$

Result:- Plancherel's Theorem

Statement:- For all $u \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$

$$\|\hat{u}\|_{L^2(\mathbb{R}^n)} = \|\check{u}\|_{L^2(\mathbb{R}^n)} = \|u\|_{L^2(\mathbb{R}^n)}$$

Property of fourier Transformation:-

1. $\forall\, u, v \in L^2(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} u\bar{v}\,dx = \int_{\mathbb{R}^n} \hat{u}\,\bar{\hat{v}}\,dx$$

Solⁿ:- Let us consider $u, v \in L^2(\mathbb{R}^n)$ and take $\alpha \in \mathbb{C}$.

then by Plancherel's theorem

$\|\widehat{u+\alpha v}\|_{L^2(\mathbb{R}^n)} = \|u+\alpha v\|_{L^2(\mathbb{R}^n)}$

Squaring on both sides, we have

$$\Rightarrow \|\widehat{u+\alpha v}\|^2_{L^2(\mathbb{R}^n)} = \|u+\alpha v\|^2_{L^2(\mathbb{R}^n)}$$

$$\Rightarrow \|\hat{u}+\alpha\hat{v}\|^2_{L^2(\mathbb{R}^n)} = \|u+\alpha v\|^2_{L^2(\mathbb{R}^n)}$$

$$\Rightarrow \int_{\mathbb{R}^n} (\hat{u}+\alpha\hat{v})\overline{(\hat{u}+\alpha\hat{v})}\,dx = \int_{\mathbb{R}^n} (u+\alpha v)\overline{(u+\alpha v)}\,dx$$

$$\Rightarrow \int_{\mathbb{R}^n} \left[|\hat{u}|^2 + |\alpha\hat{v}|^2 + \alpha\bar{\hat{u}}\hat{v} + \hat{u}\overline{(\alpha\hat{v})}\right]dx$$

$$= \int_{\mathbb{R}^n} \left[|u|^2 + |\alpha v|^2 + \bar{u}(\alpha v) + u(\overline{\alpha v})\right]dx \quad \text{---} (1)$$

$$\left\{ \begin{aligned} \|u\|_{L^2} &= \left[\int_{\mathbb{R}^n} |u|^2\right]^{1/2} \\ \|u\|^2_{L^2} &= \int_{\mathbb{R}^n} |u|^2 \\ &= \int_{\mathbb{R}^n} u\bar{u} \\ &[\,|z|^2 = z\bar{z}\,] \end{aligned} \right.$$

Since $u, v \in L^2(\mathbb{R}^n)$

$\therefore$ By apply Plancherel's thm on u, we have

$$\|u\| = \|\hat{u}\|$$

$$\Rightarrow \|u\|^2 = \|\hat{u}\|^2$$

$$\Rightarrow \int_{\mathbb{R}^n} |u|^2 dx = \int_{\mathbb{R}^n} |\hat{u}|^2 dx$$

Similarly, $\int_{\mathbb{R}^n} |\alpha v|^2 dx = \int_{\mathbb{R}^n} |\alpha\hat{v}|^2 dx$

Then eq^n (1) becomes

$$\int_{\mathbb{R}^n} \left[\alpha\bar{\hat{u}}\hat{v} + \hat{u}\overline{(\alpha\hat{v})}\right]dx = \int_{\mathbb{R}^n} \left[\bar{u}(\alpha v) + u(\overline{\alpha v})\right]dx \quad \text{---} (2)$$

Take $\alpha = 1$ and $\alpha = i$ in eq^n (2)

for $\alpha = 1$

$$\int_{\mathbb{R}^n} \left[\bar{\hat{u}}\hat{v} + \hat{u}\bar{\hat{v}}\right]dx = \int_{\mathbb{R}^n} \left[\bar{u}v + u\bar{v}\right]dx \quad \text{---} (3)$$

For $\alpha = i$

$$\int_{\mathbb{R}^n} i\left[\bar{\hat{u}}\hat{v} - \hat{u}\bar{\hat{v}}\right]dx = \int_{\mathbb{R}^n} i\left[\bar{u}v - u\bar{v}\right]dx \quad \text{---} (4)$$

Adding eq^n (3) and (4), we have

$$\Rightarrow (1+i)\int_{\mathbb{R}^n} \bar{\hat{u}}\hat{v}\,dx + (1-i)\int_{\mathbb{R}^n} \hat{u}\bar{\hat{v}}\,dx = (1+i)\int_{\mathbb{R}^n} \bar{u}v\,dx + (1-i)\int_{\mathbb{R}^n} u\bar{v}\,dx$$

comparing the coefficient of $(1+i)$ and $(1-i)$, we get

$$\int_{\mathbb{R}^n} \overline{\hat{u}}\, \hat{v}\, dx = \int_{\mathbb{R}^n} \overline{u} v\, dx$$

$$\int_{\mathbb{R}^n} \hat{u}\, \overline{\hat{v}}\, dx = \int_{\mathbb{R}^n} u \overline{v}\, dx$$

2.) $\forall\, u \in L^2(\mathbb{R}^n)$

$$\widehat{D^\alpha u} = (iy)^\alpha \hat{u} \quad \forall \text{ multi index } \alpha \text{ s.t. } D^\alpha u \in L^2(\mathbb{R}^n)$$

Solⁿ:- Consider, $\widehat{D^\alpha u}(y) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot y} D^\alpha u(x)\, dx$

$$= \frac{(-1)^{|\alpha|}}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} D_x^\alpha (e^{-ix\cdot y}) \cdot u(x)\, dx$$

↳ Here derivative w.r.t. x is $e^{-ix\cdot y}(-ix)$ but α times lena h so $e^{-ix\cdot y}(-ix)^\alpha$

$$\left[\because D^\alpha u(x) = u(x)(-1)^{|\alpha|} D_x^\alpha\right]$$

$$= \frac{(-1)^{|\alpha|}}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot y} (-iy)^\alpha u(x)\, dx$$

$$= \frac{(-1)^{|\alpha|}(-1)^\alpha (iy)^\alpha}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot y} u(x)\, dx$$

$$\left\{\begin{array}{l} \text{For } \alpha > 0 \\ (-1)^{|\alpha|}(-1)^\alpha \Rightarrow (-1)^\alpha(-1)^\alpha \\ \qquad = (-1)^{2\alpha} \\ \qquad = 1 \end{array}\right.$$

$$= (iy)^\alpha \hat{u}(y) \quad \text{for } \alpha > 0$$

3.) $\forall\, u, v \in L^2(\mathbb{R}^n)$

$$(u * v)^\wedge = (2\pi)^{n/2} \hat{u} \hat{v}$$

where $*$ denotes convolution product

Solⁿ:- Consider $(u * v)^\wedge(y) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot y} (u * v)(x)\, dx$

$$= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot y} \left(\int_{\mathbb{R}^n} u(z) v(x-z)\, dz \right) dx$$

↳ Isko fourier transform bnana h

[By Convolution Property]

$$= \underbrace{\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n}}\, e^{-iz\cdot y}\, u(z) \underbrace{\left(\int_{\mathbb{R}^n} e^{-i(x-z)\cdot y}\, v(x-z)\, dx \right)} dz$$

$$= \int_{\mathbb{R}^n} e^{-iz\cdot y}\, u(z)\, \hat{v}(y)\, dz$$

$$= \hat{v}(y) \int_{\mathbb{R}^n} e^{-iz\cdot y}\, u(z)\, dz$$

$$= \hat{v}(y)\, (2\pi)^{n/2} \cdot \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-iz\cdot y}\, u(z)\, dz$$

$$= \hat{v}(y)\, (2\pi)^{n/2}\, \hat{u}(y)$$

$$= (2\pi)^{n/2}\, \hat{u}(y)\, \hat{v}(y)$$

Hence $(u * v)^\wedge = (2\pi)^{n/2}\, \hat{u}\, \hat{v}$

4) $\forall\, u, v \in L^2(\mathbb{R}^n)$

$u = (\hat{u})^\vee$

Sol^n:- $\forall\, u, v \in L^2(\mathbb{R}^n)$, it is easy to see that

$$\int_{\mathbb{R}^n} \check{u}\, v\, dx = \int_{\mathbb{R}^n} u\, \check{v}\, dx \qquad \int \hat{u}(x)\, g(x)\, dx$$

fourier

Replacing u by $\hat{u}$ in above eq^n, we get

$$\int_{\mathbb{R}^n} (\hat{u})^\vee\, v\, dx = \int_{\mathbb{R}^n} \hat{u}\, \check{v}\, dx$$

$$= \int_{\mathbb{R}^n} \hat{u}\, \overline{(\bar{v})^\wedge}\, dx$$

$$\because (\bar{v})^\wedge = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot y}\, \overline{v(x)}\, dx$$

Taking bar on both side

$$\overline{(\bar{v})^\wedge} = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix\cdot y}\, v(x)\, dx$$

$= \check{v}$

$$\int_{\mathbb{R}^n} (\hat{u})^\vee \bar{v}\, dx = \int_{\mathbb{R}^n} u\bar{\bar{v}}\, dx \quad \text{(By property-1)}$$

$$\int_{\mathbb{R}^n} (\hat{u})^\vee v\, dx = \int_{\mathbb{R}^n} uv\, dx$$

Compairing integrand, we have

$\Rightarrow (\hat{u})^\vee v = uv$

Cancelling v on both side, we have

$$\boxed{(\hat{u})^\vee = u}$$

Fundamental solution of heat eqn using fourier Transformation

2017

Consider a heat equation

$$\left.\begin{aligned} u_t - \Delta u &= 0 \quad \text{in } \mathbb{R}^n \times (0,\infty) \\ u &= g \quad \text{on } \mathbb{R}^n \times \{t=0\} \end{aligned}\right\} \text{——} (1)$$

Taking fourier transformation on both eqns on both side,

$$\widehat{u_t - \Delta u} = \hat{0} \quad \text{in } \mathbb{R}^n \times (0,\infty)$$

$$\hat{u} = \hat{g} \quad \text{on } \mathbb{R}^n \times \{t=0\}$$

$$\Rightarrow \hat{u}_t - \Delta\hat{u} = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty)$$

$$\hat{u} = \hat{g} \quad \text{on } \mathbb{R}^n \times \{t=0\}$$

$$\Rightarrow \hat{u}_t + y^2\hat{u} = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty) \text{——} (2)$$

$$\hat{u} = \hat{g} \quad \text{on } \mathbb{R}^n \times \{t=0\} \text{——} (3)$$

$\because \Delta u = D^2 u$

$\widehat{\Delta u} = \widehat{D^2 u}$

$= (iy)^2 \hat{u}$ (By property-2)

$= -y^2 \hat{u}$

Now $\hat{u}_t + y^2\hat{u} = 0$

$\Rightarrow \hat{u}_t = -y^2\hat{u}$

$\Rightarrow \dfrac{d\hat{u}}{\hat{u}} = -y^2 dt \qquad \left[\because \hat{u}_t = \dfrac{d\hat{u}}{dt}\right]$

Integrating both side, we have

$\log\hat{u} = -|y|^2 t + \log A$

$\Rightarrow \hat{u} = Ae^{-|y|^2 t}$

where A is some constant.

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Now for $t=0$, we have

$\hat{u} = A$

$\hat{g} = A$ (By ③)

$\Rightarrow A = \hat{g}$

Hence $\hat{u} = \hat{g}\, e^{-|y|^2 t}$ —— ④

Now take inverse fourier transformation on both side, we have

$(\hat{u})^{\vee} = (\hat{g}\, e^{-|y|^2 t})^{\vee}$

$u = (\hat{g}\,\hat{F})^{\vee}$ (Using property-4)

for some function F such that $\hat{F} = e^{-|y|^2 t}$ —— ⑤

$= \dfrac{g * F}{(2\pi)^{n/2}}$ —— ⑥

By Property-3 $(u * v)^{\wedge} = (2\pi)^{n/2}\, \hat{u}\, \hat{v}$

Taking inverse fourier transformation

$((u * v)^{\wedge})^{\vee} = (2\pi)^{n/2} (\hat{u}\hat{v})^{\vee}$

$\Rightarrow (\hat{u}\hat{v})^{\vee} = \dfrac{((u * v)^{\wedge})^{\vee}}{(2\pi)^{n/2}}$

$\Rightarrow (\hat{u}\hat{v})^{\vee} = \dfrac{(u * v)}{(2\pi)^{n/2}}$

Now from eqn ⑤

$F = (e^{-|y|^2 t})^{\vee}$

$= \dfrac{1}{(2\pi)^{n/2}} \int (e^{ix\cdot y}\, e^{-|y|^2 t}\, dy$

$= \dfrac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix\cdot y - |y|^2 t}\, dy$

$= \dfrac{1}{(2\pi)^{n/2}} \left(\dfrac{\pi}{t}\right)^{n/2} e^{-\frac{|x|^2}{4t}}$

We know that $\int_{\mathbb{R}^n} e^{ix\cdot y - |y|^2 t}\, dy = \left(\dfrac{\pi}{t}\right)^{n/2} e^{-\frac{|x|^2}{4t}}$

$= \dfrac{1}{(2t)^{n/2}}\, e^{-\frac{|x|^2}{4t}}$ —— ⑦

Put the value of eqn ⑦ in eqn ⑥, we get

$$= \frac{g * e^{\frac{\cdots}{4t}}}{(2\pi)^{n/2}(2t)^{n/2}}$$

$$= \frac{1}{(4\pi t)^{n/2}} g * e^{\frac{-|x|^2}{4t}}$$

$$= \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} g(y)\, e^{-\frac{|x-y|^2}{4t}} dy \qquad \forall x \in \mathbb{R}^n,\ t>0$$

By Convolution Property
$$\left[\because u * v = \int_{\mathbb{R}^n} u(x)\, v(x-z)\, dx\right]$$

which is required solution.

Defn:- Laplace Transformation:-

Let us consider $u(x,t)$.

The Laplace transformation of u is given by

$$u^*(x,s) = \int_0^\infty e^{-st} u(x,t)\,dt$$

and inverse Laplace transformation is given by

$$u^{**}(x,t) = \int_0^\infty e^{st} u(x,s)\,ds$$

Resolvant Equation:-

Consider a heat equation

$$\left.\begin{aligned} v_t - \Delta v &= 0 \quad \text{in } U \times (0,\infty) \\ v &= f \quad \text{on } U \times \{t=0\} \end{aligned}\right\} \quad \text{——— (1)}$$

Taking Laplace transformation on both side $v(x,t)$

$$v^*(x,s) = \int_0^\infty e^{-st} v(x,t)\,dt$$

Operate $\Delta \equiv D^2$ on both side, we have,
It means yha x ki respect me h

$$\Delta v^*(x,s) = \int_0^\infty e^{-st} \Delta v(x,t)\,dt$$

$$= \int_0^\infty \underset{\mathrm{I}}{e^{-st}}\, \underset{\mathrm{II}}{v_t(x,t)}\,dt \qquad (\text{from eq}^n (1))$$

Integrating by parts on RHS, we have

$$\Delta v^*(x,s) = \left[e^{-st} v(x,t)\right]_0^\infty - \int_0^\infty (-s)\, e^{-st} v(x,t)\,dt$$

use this eqn in eqn(8)....

$$= 0 - v(x,0) + s\int_0^\infty e^{-st} v(x,t)\,dt$$

$$= -v(x,0) + s\int_0^\infty e^{-st} v(x,t)\,dt$$

$$= -f(x) + s\int_0^\infty e^{-st} v(x,t)\,dt \quad [\text{from eq}^n\ ①]$$ when $t=0$ then $v=f$

$$= -f(x) + s v^*(x,s)$$

$\Rightarrow -\Delta v^* + s v^* = f$

Hence the Laplace transformation of function $v(x,t)$ satisfy the equation

$$-\Delta u + su = f \quad \text{——} ② \quad \text{in } U\times(0,\infty)$$

The solution of eqn ② is called Resolvant solution and eqn ② is called Resolvant Equation.

Ques: What is Resolvant solⁿ
Laplace Transformation of solⁿ of heat eq

Ques:- What is Resolvant solution?

Ans:- Laplace Transformation of solution of heat eqn

$$v_t - \Delta v = 0 \quad \text{in } U\times(0,\infty)$$

$$v = f \quad \text{on } U\times\{t=0\}$$

is called Resolvant solution

Wave equation from heat Eqn

2015 OR

2014 Solution of wave equation from solution of heat equation

Solⁿ:- Let us consider wave eqn. (c)

$$\left.\begin{aligned} u_{tt} - \Delta u &= 0 \quad \text{in } \mathbb{R}^n\times(0,\infty) \\ u &= g \\ u_t &= 0 \end{aligned}\right\} \text{on } \mathbb{R}^n\times\{t=0\} \quad \text{——} ①$$

with compact support

where g is a smooth function for odd values of n

Let us consider $u(x,t) = \hat{u}(x,-t) \ \forall\ x\in\mathbb{R}^n,\ t<0$

Here t ki value -ve ke liye bi h

i.e., $u_{tt} - \Delta u = 0$ in $\mathbb{R}^n\times\mathbb{R}$,

$u(x) = x^2, x>0$
But agar hmm $-x$ rkhe to
$u(-x) = (-x)^2 = x^2 = u(x)$
So $u(x) = x^2 \ \forall\ x\in\mathbb{R}$

Define a function

$$v(x,t) = \frac{1}{(4\pi t)^{n/2}} \int_{-\infty}^{\infty} e^{-\frac{s^2}{4t}}\, u(x,s)\, ds \qquad \text{--- (2)}$$

→ solⁿ of heat eqⁿ ko satisfy krta h ya nhi firstly check (satisfy $v_t - \Delta v = 0$)

$\forall x \in \mathbb{R}^n, t>0$

and $\lim_{t\to 0} v(x,t) = g(x)$ on $\mathbb{R}^n$

Differentiate eqn (2) w.r.t. x

$$\Delta v(x,t) = \frac{1}{(4\pi t)^{1/2}} \int_{-\infty}^{\infty} e^{-\frac{s^2}{4t}}\, \Delta u(x,s)\, ds$$

$$= \frac{1}{(4\pi t)^{1/2}} \int_{-\infty}^{\infty} \underset{\mathrm{I}}{e^{-\frac{s^2}{4t}}}\, \underset{\mathrm{II}}{u_{tt}}\, ds \qquad \text{(from eqn (1))}$$

Integrating by parts on RHS, we have

$$\Delta v(x,t) = \frac{1}{(4\pi t)^{1/2}} \left[e^{-\frac{s^2}{4t}} u_t(x,s) \right]_{-\infty}^{\infty} - \frac{1}{(4\pi t)^{1/2}} \int_{-\infty}^{\infty} \left(\frac{-2s}{4t}\right) e^{-\frac{s^2}{4t}} u_t(x,s)\, ds$$

$$= \frac{1}{(4\pi t)^{1/2}} (0-0) + \frac{1}{(4\pi t)^{1/2}} \cdot \frac{1}{2t} \int_{-\infty}^{\infty} s e^{-\frac{s^2}{4t}} u_t(x,s)\, ds$$

$$= \frac{1}{2t(4\pi t)^{1/2}} \int_{-\infty}^{\infty} \underbrace{s e^{-\frac{s^2}{4t}}}_{\mathrm{I}}\, \underbrace{u_t(x,s)}_{\mathrm{II}}\, ds$$

Integrating by parts on RHS, we have

$$\Delta v(x,t) = \frac{1}{(4\pi t)^{1/2}} \left[\frac{1}{2t} \left\{ \underbrace{\left[s e^{-\frac{s^2}{4t}} u(x,s) \right]_{-\infty}^{\infty}}_{0} - \int_{-\infty}^{\infty} \left[e^{-\frac{s^2}{4t}} - \frac{2s^2}{4t} e^{-\frac{s^2}{4t}} \right] u(x,s)\, ds \right. \right.$$

(Put limit, then it is zero)

$$\Delta v(x,t) = \frac{1}{(4\pi t)^{1/2}} \left[\frac{1}{2t} \int_{-\infty}^{\infty} \left(\frac{s^2}{2t} - 1 \right) e^{-\frac{s^2}{4t}} u(x,s)\, ds \right] \qquad \text{--- (3)}$$

Now, differentiate eqn (2) w.r.t. 't', we get { In eqⁿ (2), two funⁿ h t ke so we apply product rule

$$v_t(x,t) = \frac{1}{(4\pi t)^{1/2}} \int_{-\infty}^{\infty} e^{-\frac{s^2}{4t}} \left(\frac{s^2}{4t^2} \right) u(x,s)\, ds + \frac{1}{(4\pi)^{1/2}} \left(\frac{-1}{2t^{3/2}} \right) \int_{-\infty}^{\infty} e^{-\frac{s^2}{4t}} u(x,s)\, ds$$

$$= \frac{1}{(4\pi t)^{1/2}} \int_{-\infty}^{\infty} \left[\frac{s^2}{4t^2} - \frac{1}{2t} \right] e^{-\frac{s^2}{4t}} u(x,s)\, ds$$

$$\Delta v(x,t) = \frac{1}{(4\pi t)^{1/2}} \left[\frac{1}{2t} \int_{-\infty}^{\infty} \left(\frac{s^2}{2t} - 1 \right) e^{\frac{-s^2}{4t}} u(x,s)\,ds \right] \quad \text{—④}$$

Now from eqn ⑤ and ④, we get

$$v_t(x,t) = \Delta v(x,t)$$

$$\Rightarrow v_t - \Delta v = 0$$

Also from eqn ②

$$v = g \text{ on } \mathbb{R}^n \times \{t=0\}$$

Hence v defined by eqn ② satisfy heat equation

$$\left. \begin{aligned} v_t - \Delta v &= 0 \text{ in } \mathbb{R}^n \times (0,\infty) \\ v &= g \text{ on } \mathbb{R}^n \times \{t=0\} \end{aligned} \right\} \text{—⑤}$$

We know that the fundamental solution of heat equation ⑤ is given by

$$v(x,t) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)\,dy \quad \text{—⑥}$$

Now from eqn ②,

$$\frac{1}{(4\pi t)^{1/2}} \int_{-\infty}^{\infty} e^{\frac{-s^2}{4t}} u(x,s)\,ds = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)\,dy \quad \text{—⑦}$$

Let $\lambda = \frac{1}{4t}$

Then eqn ⑦ becomes

$$\frac{1}{(\pi/\lambda)^{1/2}} \int_{-\infty}^{\infty} e^{-s^2\lambda} u(x,s)\,ds = \frac{1}{(\pi/\lambda)^{n/2}} \int_{\mathbb{R}^n} e^{-\lambda|x-y|^2} g(y)\,dy$$

Since $u(x,s) = u(x,-s)$ $\quad \left\{ f(-x) = f(x) = \text{even fun}^n \right.$

$\therefore$ $u(x,s)$ is an even function of s

Also $e^{-s^2\lambda}$ is an even function of s $\quad \left\{ e^{-(-s)^2\lambda} = e^{-s^2\lambda} = \text{even fun}^n \right.$

$\therefore$ Above eqn becomes

$$2\left(\frac{\lambda}{\pi}\right)^{1/2} \int_0^{\infty} e^{-s^2\lambda} u(x,s)\,ds = \left(\frac{\lambda}{\pi}\right)^{n/2} \int_{\mathbb{R}^n} e^{-\lambda|x-y|^2} g(y)\,dy$$

$\left[\int_{-\infty}^{\infty} f(x)\,dx \right.$ If f is even then this integral is equal to $\left. 2\int_0^{\infty} f(x)\,dx \right]$

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$$\int_0^\infty e^{-s^2\lambda} u(x,s)\,ds = \frac{1}{2}\left(\frac{\lambda}{\pi}\right)^{\frac{n-1}{2}} \int_{\mathbb{R}^n} e^{-\lambda|x-y|^2} g(y)\,dy$$

We know that

$$\int_{\mathbb{R}^n} f\,dx = \int_0^\infty \left(\int_{\partial B(x,r)} f\,ds\right) dr \quad \text{[By strong maximum principle]}$$

Apply this to above eqn,

$$\int_0^\infty e^{-s^2\lambda} u(x,s)\,ds = \frac{1}{2}\left(\frac{\lambda}{\pi}\right)^{\frac{n-1}{2}} \int_0^\infty \left(\int_{\partial B(x,r)} e^{-\lambda|x-y|^2} g(y)\,ds(y)\right) dr$$

$$= \frac{1}{2}\left(\frac{\lambda}{\pi}\right)^{\frac{n-1}{2}} \int_0^\infty \left(n\alpha(n) r^{n-1} ⨍_{\partial B(x,r)} e^{-\lambda|x-y|^2} g(y)\,ds(y)\right) dr$$

$$= \frac{1}{2}\left(\frac{\lambda}{\pi}\right)^{\frac{n-1}{2}} n\alpha(n) \int_0^\infty r^{n-1} e^{-\lambda r^2} G(x,r)\,dr \quad \text{— (8)}$$

$$\left\{ \begin{array}{l} \because y \in \partial B(x,r) \\ \therefore |y-x| = r \end{array} \right.$$

(7) —

where $G(x,r) = ⨍_{\partial B(x,r)} g(y)\,ds(y)$ — (9)

(By Euler poisson Darboux thm)

Now, n is an odd natural number

$\therefore$ Put $n = 2k+1$, $k>0$

It is easy to see that $\frac{(-1)^k}{2^k}\left(\frac{1}{r}\frac{d}{dr}\right)^k e^{-\lambda r^2} = \lambda^k e^{-\lambda r^2}$ (By PMI) — (10)

↳ In k Maths we prove it.

Consider $\lambda^{\frac{n-1}{2}} \int_0^\infty e^{-\lambda r^2} r^{n-1} G(x,r)\,dr \left(= \int_0^\infty r^{2k} \lambda^k e^{-\lambda r^2} G(x,r)\,dr\right)$

↳ Put $n = 2k+1$

Put the LHS of (10) in above eqn, we get

$$\lambda^{\frac{n-1}{2}} \int_0^\infty e^{-\lambda r^2} r^{n-1} G(x,r)\,dr = \frac{(-1)^k}{2^k} \int_0^\infty \left[\left(\frac{1}{r}\frac{d}{dr}\right)^k e^{-\lambda r^2}\right] r^{2k} G(x,r)\,dr$$

$$= \frac{(-1)^k}{2^k} \int_0^\infty r \left[\left(\frac{1}{r}\frac{d}{dr}\right)^k (e^{-\lambda r^2})\right] (r^{2k-1} G(x,r))\,dr$$

Use this eqn in eqn (8), we get

$$\int_0^\infty u(x,s)\, e^{-\lambda s^2} ds = \frac{(-1)^K n\alpha(n)}{\pi^{\frac{n-1}{2}}\, 2^{K+1}} \int_0^\infty r\left(\frac{1}{r}\frac{d}{dr}\right)^K \left(r^{(2K-1)} G(x,r)\right) e^{-\lambda r^2} dr$$

Putting $n = 2K+1$

$$= \frac{(-1)^K n\alpha(n)}{\pi^K 2^{K+1}} \int_0^\infty r\left(\frac{1}{r}\frac{d}{dr}\right)^K \left(r^{2K-1} G(x,r)\right) e^{-\lambda r^2} dr$$

Taking $\tau = s^2$ and $\tau = r^2$ on R.H.S and R.H.S. respectively, we see that both are Laplace transformation of some functions. Both are equal iff there original functions are identical.

i.e, $u(x,t) = \frac{(-1)^K n\alpha(n)}{\pi^K 2^{K+1}}\, t\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^K \left(t^{2K-1} G(x,t)\right)$ ——— (11)

Now, for $n = 2K+1$,

$$\alpha(n) = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)} = \frac{\pi^{\frac{2K+1}{2}}}{\Gamma\left(\frac{2K+1}{2}+1\right)} = \frac{\pi^{\frac{2K+1}{2}}}{\Gamma\left(\frac{n}{2}+1\right)}$$

$$\Rightarrow \frac{n\alpha(n)}{\pi^K 2^{K+1}} = \frac{n\pi^{\frac{2K+1}{2}}}{\pi^K 2^{K+1}\Gamma\left(\frac{n}{2}+1\right)}$$

$$= \frac{n\pi^{\frac{1}{2}}}{2^{K+1}\left[\frac{n}{2}\left(\frac{n}{2}-1\right)\cdots\frac{3}{2}\cdot\frac{1}{2}\right]\Gamma\left(\frac{1}{2}\right)} \qquad [\Gamma(n) = (n-1)\Gamma(n-1)]$$

$$= \frac{n\pi^{1/2}}{2^{K+1}\left[\frac{n}{2}\left(\frac{n}{2}-1\right)\cdots\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}\right]} \qquad \left[\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$

$1, 3, 5 \cdots (n-2), n$

$T_{p\text{th}} = 1 + (p-1)2$

$n = 1 + 2p - 2$

$n = 2p - 1$

$\frac{n+1}{2} = p$

$\frac{2K+1+1}{2} = p$

$\boxed{p = K+1}$

$$= \frac{n}{2^{K+1}\left[\frac{n}{2}\left(\frac{n}{2}-1\right)\cdots\frac{3}{2}\cdot\frac{1}{2}\right]}$$

$$= \frac{n}{2^{K+1}\left[\frac{n(n-2)(n-4)\cdots 5.3.1}{2^{K+1}}\right]}$$

$$= \frac{1}{1.3.5\cdots(n-4)(n-2)}$$

$$= \frac{1}{\gamma_n} \quad \text{(say)}$$

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Use this value in eqn (11), we have

$$u(x,t) = \frac{(-1)^k}{\gamma_n} t \left[\left(\frac{1}{t} \frac{d}{dt} \right)^k \left(t^{2k-1} G(x,t) \right) \right]$$

$$= \frac{(-1)^k}{\gamma_n} t \left[\left(\frac{1}{t} \frac{d}{dt} \right) \left(\frac{1}{t} \frac{d}{dt} \right)^{k-1} \left(t^{2k-1} G(x,t) \right) \right]$$

$$= \frac{(-1)^k}{\gamma_n} \frac{d}{dt} \left[\left(\frac{1}{t} \frac{d}{dt} \right)^{k-1} \left(t^{2k-1} \fint_{\partial B(x,t)} g(y)\, ds(y) \right) \right] \quad \text{(from eq}^n \text{ (9))}$$

$$= \frac{(-1)^{\frac{n-1}{2}}}{\gamma_n} \frac{d}{dt} \left[\left(\frac{1}{t} \frac{d}{dt} \right)^{\frac{n-3}{2}} \left(t^{n-2} \fint_{\partial B(x,t)} g(y)\, ds(y) \right) \right] \quad \text{——— (12)}$$

Eqn (12) is the required solution of eqn (1)

$$\left\{ \begin{array}{l} n = 2k+1 \Rightarrow k = \frac{n-1}{2} \\ k-1 = \frac{n-1}{2} - 1 \Rightarrow \frac{n-3}{2} \end{array} \right.$$