6 **Canonical Product :—** By Weierstrass Factorization theorem, $\exists$ an entire function which prescribed arbitrary zero provided that in the case of infinitely many zeros $z_n \to \infty$ as $n \to \infty$.

$\therefore$ Every entire function with these zeros can be written in the form

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) \exp\left(\frac{z}{z_n} + \frac{1}{2}\left(\frac{z}{z_n}\right)^2 + \dots + \frac{1}{m_n}\left(\frac{z}{z_n}\right)^{m_n}\right) \quad -(1)$$

where $z_n \neq 0$ and the product is taken over all $z_n \neq 0$ and m_n are certain integers and $g(z)$ is an entire function.

Then the proof of Weierstrass factorisation theorem shows that the infinite product

$$\prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) \exp\left(\frac{z}{z_n} + \frac{1}{2}\left(\frac{z}{z_n}\right)^2 + \dots \frac{1}{h}\left(\frac{z}{z_n}\right)^h\right) = \prod_{n=1}^{\infty} E\left(\frac{z}{z_n}, h\right) \quad (2)$$

and it converges and represents an entire function.

If the series $\sum_{n=1}^{m} \left(\frac{R}{|z_n|}\right)^{h+1}$ converges for all R

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ie if $\sum_{n=1}^{m} \frac{1}{|z_n|^{h+1}} < \infty$

Assume that h is the smallest integer for which the series converges. Then we say that ② is the canonical product associated with the sequence $\{z_n\}$ and h is called the genus of the canonical product.

Note:- If in ①, $g(z)$ reduces to a polynomial then we say that $f(z)$ is of finite genus and we define the genus of $f(z)$ to be the degree of this polynomial or to be the genus of the canonical product whichever is greater.

Jensen's Formula :- Let $f(z)$ be an analytic function in the closed disc $|z| \leq R$; assume $f(0) \neq 0$ and $f(z) \neq 0$ on $|z| = R$. If $z_1, z_2, \ldots z_n$ are the zeros of $f(z)$ in the open disc $|z| < R$; each repeated as often as its multiplicity then prove that

$$\log|f(0)| = -\sum_{i=1}^{n} \log\left(\frac{R}{|z_i|}\right) + \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\phi})|\, d\phi$$

Proof:- Consider the function

$$F(z) = f(z) \prod_{i=1}^{n} \frac{R^2 - \bar{z}_i z}{R(z - z_i)} \quad \text{--- ①}$$

Clearly $F(z)$ is analytic in any domain in which $f(z)$ is analytic and also $F(z) \neq 0$ for $|z| \leq R$

Gauss Mean Value Theorem :- If $f(z)$ is analytic in a domain D and if the circular domain $|z-z_0| \le \rho$ is contained in D, then $f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0+\rho e^{i\theta})\, d\theta$.

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Therefore $F(z)$ is analytic and never zero (non-zero) on an open disc $|z|<\rho$ for some $\rho > R$

Now

$$|F(z)| = \left| \prod_{i=1}^{n} \frac{R^2-\bar{z}_i z}{R(z-z_i)} f(z) \right|$$

$$= \left| \prod_{i=1}^{n} \frac{R^2-\bar{z}_i z}{R(z-z_i)} \right| |f(z)| \quad -(2)$$

Now on $|z| = R$, $z = Re^{i\phi}$ $\quad -(3)$

$$|F(z)| = \prod_{i=1}^{n} \left| \frac{R^2-\bar{z}_i Re^{i\phi}}{R(Re^{i\phi}-z_i)} \right| |f(z)|$$

$$= |f(z)| \prod_{i=1}^{n} \left| \frac{R(R-\bar{z}_i e^{i\phi})}{Re^{i\phi}(R-z_i e^{-i\phi})} \right|$$

$$= |f(z)| \prod_{i=1}^{n} \left| \frac{R-\bar{z}_i e^{i\phi}}{e^{i\phi}(R-z_i e^{-i\phi})} \right|$$

$$= |f(z)|$$

[$\because$ $R-z_i e^{-i\phi}$ & $R-\bar{z}_i e^{i\phi}$ are conjugates and modulus of conjugates are same and $|e^{i\phi}| = 1$

$\Rightarrow$ $|F(z)| = |f(z)|$ $\quad -(4)$

Since $F(z)$ is analytic and non-zero in $|z|<\rho$,

$\log|F(z)|$ is analytic in $|z|<\rho$

$\Rightarrow$ Real part of $\log|F(z)|$ is harmonic there (in this region)

$\therefore$ By Gauss mean value theorem for $\log|F(z)|$

$$\log|F(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|F(Re^{i\phi})|\, d\phi \quad -(5)$$

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From ①, we have

$$F(0) = f(0)\prod_{i=1}^{n}\left(\frac{R^2}{-Rz_i}\right) = f(0)\prod_{i=1}^{n}\left(\frac{-R}{z_i}\right)$$

$$|F(0)| = \left| f(0)\prod_{i=1}^{n}\left(\frac{-R}{z_i}\right)\right| = |f(0)|\prod_{i=1}^{n}\frac{R}{|z_i|}$$

$$\log|F(0)| = \log|f(0)| + \sum_{i=1}^{n}\log\frac{R}{|z_i|} \qquad -⑥$$

By eqn. ④

$$|F(Re^{i\phi})| = |f(Re^{i\phi})| \qquad -⑦$$

From eqn. ⑤, we have

$$\log|F(0)| = \frac{1}{2\pi}\int_0^{2\pi}\log|F(Re^{i\phi})|\,d\phi$$

$$\log|f(0)| + \sum_{i=1}^{n}\log\frac{R}{|z_i|} = \frac{1}{2\pi}\int_0^{2\pi}\log|f(Re^{i\phi})|\,d\phi$$

[using ⑥ & ⑦]

$$\Rightarrow \log|f(0)| = -\sum_{i=1}^{n}\log\frac{R}{|z_i|} + \frac{1}{2\pi}\int_0^{2\pi}\log|f(Re^{i\phi})|\,d\phi \qquad ⑧$$

Corollary - More generally let $g(z)$ also satisfy the hypothesis of the above theorem with the zeros in $|z|<R$ at $w_1, w_2, \ldots w_m$ and $h(z) = \frac{f(z)}{g(z)}$. Then

$$\log|h(0)| = \sum_{j=1}^{m}\log\frac{R}{|w_j|} - \sum_{i=1}^{n}\log\frac{R}{|z_i|} + \frac{1}{2\pi}\int_0^{2\pi}\log|h(Re^{i\phi})|\,d\phi$$

Proof :- Applying above theorem to $f(z)$ and $g(z)$, we have

$$\log|f(0)| = -\sum_{i=1}^{n} \log \frac{R}{|z_i|} + \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\phi})|\, d\phi \quad -(9)$$

and

$$\log|g(0)| = -\sum_{j=1}^{m} \log \frac{R}{|w_j|} + \frac{1}{2\pi}\int_0^{2\pi} \log|g(Re^{i\phi})|\, d\phi \quad -(10)$$

Subtracting (10) from (9), we have

$$\log|f(0)| - \log|g(0)| = -\sum_{i=1}^{n} \log \frac{R}{|z_i|} + \sum_{j=1}^{m} \log \frac{R}{|w_j|} + \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\phi})|\, d\phi - \frac{1}{2\pi}\int_0^{2\pi} \log|g(Re^{i\phi})|\, d\phi$$

$$\Rightarrow \log\left|\frac{f(0)}{g(0)}\right| = \sum_{j=1}^{m} \log \frac{R}{|w_j|} - \sum_{i=1}^{n} \log \frac{R}{|z_i|} + \frac{1}{2\pi}\int_0^{2\pi} \log\left|\frac{f(Re^{i\phi})}{g(Re^{i\phi})}\right| d\phi$$

$$\Rightarrow \log|h(0)| = \sum_{j=1}^{m} \log \frac{R}{|w_j|} - \sum_{i=1}^{n} \log \frac{R}{|z_i|} + \frac{1}{2\pi}\int_0^{2\pi} \log|h(Re^{i\phi})|\, d\phi$$

Hadamard's three circle theorem :- If $f(z)$ is analytic in the closed ring $r_1 \le |z| \le r_3$ and if $r_1 < r_2 < r_3$ and also M_1, M_2 and M_3 are respectively the maximas of $|f(z)|$ on the three circles $|z| = r_1$, $|z| = r_2$ and $|z| = r_3$ then we have to prove that

$$M_2^{\log\left(\frac{r_3}{r_1}\right)} \le M_1^{\log\left(\frac{r_3}{r_2}\right)} \cdot M_3^{\log\left(\frac{r_2}{r_1}\right)}$$

Proof Let us consider the function

$$g(z) = z^k f(z) \quad -(1)$$

where k is a real constant.

Since $f(z)$ is analytic in the annulus between $|z| = r_1$ and $|z| = r_3$

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Therefore $g(z)$ is analytic in the annulus between $|z| = r_1$ and $|z| = r_3$

$\therefore$ $|g(z)|$ occurs on one of the boundary of circles so that

$$|g(z)| \leq \max\{r_1^k M_1, r_3^k M_3\} \quad \forall z \text{ with } r_1 \leq |z| \leq r_3 \quad -②$$

i.e $$|z^k f(z)| \leq \max\{r_1^k M_1, r_3^k M_3\} \quad -③$$

Now if $r_1 < r_2 < r_3$ then

$$r_2^k M_2 \leq \max\{r_1^k M_1, r_3^k M_3\} \quad -Ⓐ$$

Since k is at our choice.

So, we choose k s.t.

$$r_1^k M_1 = r_3^k M_3$$

Taking log on both side, we get

$$k \log r_1 + \log M_1 = k \log r_3 + \log M_3$$

$$\Rightarrow k(\log r_1 - \log r_3) = \log M_3 - \log M_1$$

$$\Rightarrow k \log\left(\frac{r_1}{r_3}\right) = \log\left(\frac{M_3}{M_1}\right)$$

$$\Rightarrow k = \frac{\log\left(\frac{M_3}{M_1}\right)}{\log\left(\frac{r_1}{r_3}\right)} = \frac{-\log\left(\frac{M_3}{M_1}\right)}{\log\left(\frac{r_3}{r_1}\right)}$$

Now, from Ⓐ we have

$$r_2^k M_2 \leq r_1^k M_1$$

$$\Rightarrow M_2 \leq \left(\frac{r_1}{r_2}\right)^k M_1 = \left(\frac{r_2}{r_1}\right)^{-k} M_1 \quad -⑤$$

Taking $\left(\log\left(\frac{r_3}{r_1}\right)\right)$ th power, we get

$$M_2^{\log\left(\frac{r_3}{r_1}\right)} \leq \left(\frac{r_2}{r_1}\right)^{-k\log\left(\frac{r_3}{r_1}\right)} M_1^{\log\left(\frac{r_3}{r_1}\right)} \quad -⑥$$

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$$M_2^{\log\left(\frac{r_3}{r_1}\right)} \le \left(\frac{r_2}{r_1}\right)^{\log\left(\frac{M_3}{M_1}\right)} M_1^{\log\left(\frac{r_3}{r_1}\right)} \quad - (I)$$

[using (4)]

since $a^{\log b} = \left(e^{\log a}\right)^{\log b}$
$= e^{\log a \cdot \log b} = \left(e^{\log b}\right)^{\log a}$
$= b^{\log a}$

$\Rightarrow \quad a^{\log b} = b^{\log a}$

Using in (I), we get

$$M_2^{\log\left(\frac{r_3}{r_1}\right)} \le \left(\frac{M_3}{M_1}\right)^{\log\left(\frac{r_2}{r_1}\right)} M_1^{\log\left(\frac{r_3}{r_1}\right)}$$

$$= (M_3)^{\log\left(\frac{r_2}{r_1}\right)} M_1^{-\log\left(\frac{r_2}{r_1}\right)} \cdot M_1^{\log\left(\frac{r_3}{r_1}\right)}$$

$$= M_3^{\log\left(\frac{r_2}{r_1}\right)} M_1^{\log\frac{r_3}{r_1} - \log\frac{r_2}{r_1}}$$

$$\Rightarrow \quad M_2^{\log\left(\frac{r_3}{r_1}\right)} \le M_3^{\log\left(\frac{r_2}{r_1}\right)} M_1^{\log\left(\frac{r_3}{r_2}\right)}$$

H.P.

Remark By Hadamard's three circle theorem, we have

$$M_2^{\log\left(\frac{r_3}{r_1}\right)} \le M_3^{\log\left(\frac{r_2}{r_1}\right)} M_1^{\log\left(\frac{r_3}{r_2}\right)}$$

Taking log on both sides, we get

$$\log M_2^{\log\left(\frac{r_3}{r_1}\right)} \le \log M_3^{\log\left(\frac{r_2}{r_1}\right)} + \log M_1^{\log\left(\frac{r_3}{r_2}\right)}$$

$$\Rightarrow \log\left(\frac{r_3}{r_1}\right) \cdot \log M_2 \le \log\left(\frac{r_2}{r_1}\right) \cdot \log M_3 + \log\left(\frac{r_3}{r_2}\right) \cdot \log M_1$$

$$\Rightarrow \log M_2 \le \frac{\log\left(\frac{r_2}{r_1}\right) \log M_3}{\log\left(\frac{r_3}{r_1}\right)} + \frac{\log\left(\frac{r_3}{r_2}\right) \log M_1}{\log\left(\frac{r_3}{r_1}\right)}$$

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$$\Rightarrow \log M_2 \le \frac{(\log r_2 - \log r_1)}{(\log r_3 - \log r_1)} \log M_3 + \frac{(\log(r_3) - \log r_2)}{(\log r_3 - \log r_1)} \log M_1$$

which shows that $\log M_r$ is a convex function.

Poisson's Jensen Formula :— If $f(z)$ is analytic in the closed disc $|z| \le R$ and assume that $f(z) \neq 0$ on $|z| = R$. If $z_1, z_2, \dots, z_n$ are the zeros of $f(z)$ in the open disc $|z| < R$ each repeated as often as in its multiplicity and $z = re^{i\Phi}$; $0 < r < R$ then prove that

$$\log|f(z)| = -\sum_{i=1}^{n} \log\left(\frac{R^2 - \bar{z}_i z}{R(z - z_i)}\right) + \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)\log|f(Re^{i\phi})|}{R^2 + r^2 - 2Rr\cos(\theta - \phi)}\, d\phi$$

Proof :- Consider

$$F(z) = f(z) \prod_{i=1}^{n} \frac{R^2 - \bar{z}_i z}{R(z - z_i)} \quad — (1)$$

Clearly $F(z)$ is analytic in any domain in which the function $f(z)$ is analytic and also $F(z) \neq 0$ for $|z| \le R$

$\therefore$ $F(z)$ is analytic and never zero on an open disc $|z| < \rho$ for some $\rho > R$

Taking modulus on both sides of (1), we have

$$|F(z)| = \left| f(z) \prod_{i=1}^{n} \frac{R^2 - \bar{z}_i z}{R(z - z_i)} \right|$$

$$= |f(z)| \left| \prod_{i=1}^{n} \frac{R^2 - \bar{z}_i z}{R(z - z_i)} \right| \quad — (2)$$

Poisson's Integral Formula:- If $f(z)$ is analytic within and on a circle C defined by $|z|=R$ and if a is any point within C, then

$$f(a) = \frac{1}{2\pi i}\int_C \frac{(R^2-a\bar{a})f(z)\,dz}{(z-a)(R^2-z\bar{a})}.$$ Hence deduce the Poisson formula

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Now on $|z|=R$, $z=Re^{i\phi}$

$$\Rightarrow |F(z)| = |f(z)|\left|\prod_{i=1}^{n}\frac{R^2-\bar{z}_i Re^{i\phi}}{R(Re^{i\phi}-z_i)}\right|$$

$$= |f(z)|\left|\prod_{i=1}^{n}\frac{R-\bar{z}_i e^{i\phi}}{e^{i\phi}(R-z_i e^{-i\phi})}\right| \quad -(3)$$

Since $R-\bar{z}_i e^{i\phi}$ and $R-z_i e^{-i\phi}$ are conjugates of each other and we know that modulus of conjugates are same and $|e^{i\phi}|=1$

So eqn. (3) becomes

$$|F(z)| = |f(z)| \quad -(4)$$

Now, since $F(z)$ is analytic and non-zero in the open disc $|z|<\rho$; $\rho>R$

$\therefore$ $\log|F(z)|$ is harmonic in region $|z|<\rho$

So, by Poisson integral formula

$$\log|F(z)| = \frac{1}{2\pi}\int_0^{2\pi}\frac{(R^2-r^2)\log|F(Re^{i\phi})|}{R^2-2Rr\cos(\theta-\phi)+r^2}\,d\phi \quad -(5)$$

Also $$|F(z)| = |f(z)|\prod_{i=1}^{n}\left|\frac{R^2-\bar{z}_i z}{R(z-z_i)}\right|$$ [From (2)]

Taking log. on both side

$$\log|F(z)| = \log|f(z)| + \sum_{i=1}^{n}\log\left|\frac{R^2-\bar{z}_i z}{R(z-z_i)}\right| \quad -(6)$$

From (4), we have

$$\log|F(Re^{i\phi})| = \log|f(Re^{i\phi})| \quad -(7)$$

From (5) and (6), we have

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2-r^2)\, f(Re^{i\phi})\, d\phi}{R^2-2Rr\cos(\theta-\phi)+r^2}$$

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where $a = re^{i\theta}$ is any point insider the circle $|z| = R$.

$$\log|f(z)| + \sum_{i=1}^{n} \log\left|\frac{R^2-\bar{z}_i z}{R(z-z_i)}\right| = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2-r^2)\log|F(Re^{i\phi})|\, d\phi}{R^2-2Rr\cos(\theta-\phi)+r^2}$$

$$\log|f(z)| = -\sum_{i=1}^{n} \log\left|\frac{R^2-\bar{z}_i z}{R(z-z_i)}\right| + \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2-r^2)\log|f(Re^{i\phi})|\, d\phi}{R^2-2Rr\cos(\theta-\phi)+r^2}$$

[using (1)]

<u>Generalisation of theorem</u> :— If $g(z)$ also satisfies the hypothesis of Poisson's Jensen formula with zeros $w_1, w_2, \ldots, w_m$ in $|z|<R$ and $\frac{f(z)}{g(z)} = h(z)$ then prove that

$$\log|h(z)| = \sum_{j=1}^{m} \log\left|\frac{R^2-z\bar{w}_j}{R(z-w_j)}\right| - \sum_{i=1}^{n} \log\left|\frac{R^2-z\bar{z}_i}{R(z-z_i)}\right| + \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2-r^2)\log|h(Re^{i\phi})|\, d\phi}{R^2-2Rr\cos(\theta-\phi)+r^2}$$

<u>Proof</u> :- By Poisson's Jensen Formula, we have

$$\log|f(z)| = -\sum_{i=1}^{n} \log\left|\frac{R^2-\bar{z}_i z}{R(z-z_i)}\right| + \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2-r^2)\log|f(Re^{i\phi})|\, d\phi}{R^2-2Rr\cos(\theta-\phi)+r^2} \quad -(1)$$

and

$$\log|g(z)| = -\sum_{j=1}^{m} \log\left|\frac{R^2-z\bar{w}_j}{R(z-w_j)}\right| + \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2-r^2)\log|g(Re^{i\phi})|\, d\phi}{R^2-2Rr\cos(\theta-\phi)+r^2} \quad -(2)$$

Subtracting (2) from (1), we get

$$\log|f(z)| - \log|g(z)| = -\sum_{i=1}^{n} \log\left|\frac{R^2-\bar{z}_i z}{R(z-z_i)}\right| + \sum_{j=1}^{m} \log\left|\frac{R^2-z\bar{w}_j}{R(z-w_j)}\right|$$

$$+ \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2-r^2)\left[\log|f(Re^{i\phi})| - \log|g(Re^{i\phi})|\right] d\phi}{R^2-2Rr\cos(\theta-\phi)+r^2}$$

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$$\log\left|\frac{f(z)}{g(z)}\right| = -\sum_{i=1}^{n} \log\left|\frac{R^2-\bar{z}_i z}{R(z-z_i)}\right| + \sum_{j=1}^{m} \log\left|\frac{R^2 - z\bar{w}_j}{R(z-w_j)}\right|$$

$$+ \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2-r^2)}{R^2-2Rr\cos(\theta-\phi)+r^2} \log\left|\frac{f(Re^{i\phi})}{g(Re^{i\phi})}\right| d\phi$$

$$\Rightarrow \log|h(z)| = \sum_{j=1}^{m} \log\left|\frac{R^2-z\bar{w}_j}{R(z-w_j)}\right| - \sum_{i=1}^{n} \log\left|\frac{R^2-\bar{z}_i z}{R(z-z_i)}\right| + \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2-r^2)\log|h(Re^{i\phi})|}{R^2-2Rr\cos(\theta-\phi)+r^2} d$$