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UNIT–III [Half Part]

Fourier Transform: → Fourier transform of a function $f(x)$ is given by

$$\boxed{f(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)\, e^{-ikx}\, dx}$$

Also, inverse fourier transform $f(x)$ is given by,

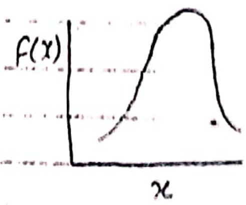
$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F(k)\, e^{+ikx}\, dk}$$

It is also called as fourier inversion formula.

Fourier transform of $f(x) = e^{-x^2/2}$: →

The function $f(x) = e^{-x^2/2}$ is a Guassian function. Fourier transform of this function is given by

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)\, e^{-ikx}\, dx \quad \text{— (I)}$$



Guissionn function

$$f(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-x^2/2}\, e^{-ikx}\, dx$$

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\left(\frac{x^2}{2} + ikx\right)}\, dx$$

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\left[\left(\frac{x}{\sqrt{2}}\right)^2 + \left(\frac{ik}{\sqrt{2}}\right)^2 + 2\frac{ik}{\sqrt{2}}\frac{x}{\sqrt{2}}\right]}\, e^{-k^2/2}\, dx$$

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\left(\frac{x}{\sqrt{2}} + \frac{ik}{\sqrt{2}}\right)^2}\, e^{-k^2/2}\, dx$$

$$F(k) = \frac{e^{-k^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\left(\frac{x}{\sqrt{2}} + \frac{ik}{\sqrt{2}}\right)^2}\, dx$$

$$F(k) = \frac{e^{-k^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\left(\frac{x}{\sqrt{2}} + \frac{ik}{\sqrt{2}}\right)^2} dx$$

let $\frac{x}{\sqrt{2}} + \frac{ik}{\sqrt{2}} = y$

$\therefore \frac{dx}{\sqrt{2}} = dy \Rightarrow dx = \sqrt{2}\, dy$

also,

as $x \to -\infty \Rightarrow y \to -\infty$

$x \to +\infty \Rightarrow y \to +\infty$

$$F(k) = \frac{e^{-k^2/2}}{\sqrt{2\pi}} \cdot \sqrt{2} \int_{-\infty}^{+\infty} e^{-y^2} dy$$

Now e^{-y^2} is an even function of y, By using property of even function,

$$F(k) = \frac{e^{-k^2/2}}{\sqrt{2\pi}} \sqrt{2} \cdot 2 \int_{0}^{\infty} e^{-y^2} dy$$

$\left[\because \text{If } f(x) \text{ is even fun. } \int_{-\infty}^{+\infty} f(x)\,dx = 2\int_{0}^{\infty} f(x)\,dx\right]$

let $y^2 = z$

$\Rightarrow 2y\,dy = dz$

$$dy = \frac{dz}{2y} = \frac{dz}{2 \cdot z^{1/2}}$$

Also as $y \to 0 \Rightarrow z \to 0$

$y \to \infty, \Rightarrow z \to \infty$

$$F(k) = \frac{e^{-k^2/2}}{\sqrt{2\pi}} \sqrt{2} \cdot \frac{\cancel{2}}{\cancel{2}} \int_{0}^{\infty} e^{-z} z^{-1/2} dz$$

$$\Rightarrow F(k) = \frac{e^{-k^2/2}}{\sqrt{2\pi}} \sqrt{2} \int_0^{\infty} e^{-z} z^{-1/2} dz$$

Now $\int_0^{\infty} e^{-z} z^{-1/2}$ = Gamma function of $1/2$
$= \Gamma(\frac{1}{2}) = \sqrt{\pi}$ (By definition of Gamma fun.)

$$\therefore F(k) = \frac{e^{-k^2/2}}{\cancel{\sqrt{2\pi}}} \cancel{\sqrt{2}} \cdot \cancel{\sqrt{\pi}}$$

$$\Rightarrow \boxed{F(k) = e^{-k^2/2}}$$

Which also shows that Fourier transform of Guassian function is also a Guassian function.

Fourier transform of function $f(x) = 1$ for $|x| < a$
$= 0$ for $|x| > a$ $\rightarrow$

The function $f(x)$ is given by,

$$f(x) = 1 \quad \text{for } |x| < a$$
$$= 0 \quad \text{for } |x| > a \quad \text{(I)}$$

Now, Fourier transform of $f(x)$ is given by

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)\, e^{-ikx} dx$$

$$\Rightarrow F(k) = \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{-a} f(x) e^{-ikx} dx + \int_{-a}^{a} f(x) e^{-ikx} dx + \int_{a}^{\infty} f(x) e^{-ikx} dx \right]$$

$$F(k) = \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{-a} 0 \cdot e^{-ikx} dx + \int_{-a}^{a} 1 \cdot e^{-ikx} dx + \int_{a}^{\infty} 0 \cdot e^{-ikx} dx \right]$$

$$F(k) = \frac{1}{\sqrt{2\pi}}\left[\int_{-a}^{+a} e^{-ikx}\,dx\right]$$

$$F(k) = \frac{1}{\sqrt{2\pi}}\left[\frac{e^{-ikx}}{-ik}\right]_{-a}^{+a}$$

$$\Rightarrow F(k) = \frac{1}{\sqrt{2\pi}}\left[\frac{e^{-ika} - e^{ika}}{-ik}\right]$$

$$\Rightarrow F(k) = \frac{1}{\sqrt{2\pi}}\left[\frac{e^{ika} - e^{-ika}}{ik}\right]$$

$$\Rightarrow F(k) = \frac{2}{\sqrt{2\pi}\cdot k}\cdot\frac{e^{ika} - e^{-ika}}{2i}$$

$$\Rightarrow \boxed{F(k) = \sqrt{\frac{2}{\pi}}\,\frac{\sin ka}{k}}$$ where $\frac{e^{ika} - e^{-ika}}{2i} = \sin ka$

Properties of Fourier transform :→

(i) Addition theorem :→ This theorem states that if $F(k)$ and $G(k)$ are respectively Fourier transform of $f(x)$ and $g(x)$, then $[\alpha F(k) + \beta G(k)]$ is Fourier transform of function $[\alpha f(x) + \beta g(x)]$, where α and β are constants.

Proof :→ Since $F(k)$ is Fourier transform of $f(x)$

$$\therefore F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-ikx} f(x)\,dx \quad \text{—— (I)}$$

Similarly $G(k)$ is Fourier transform of $g(x)$,

$$\therefore G(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-ikx} g(x)\,dx \quad \text{—— (II)}$$

$$\alpha F(k) + \beta G(k) = \frac{\alpha}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ikx} f(x)\,dx + \frac{\beta}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ikx} g(x)\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ikx} (\alpha f(x) + \beta g(x))\,dx$$

which means $\alpha F(k) + \beta G(k)$ is fourier transform of $\alpha f(x) + \beta g(x)$.

(ii) Shifting theoram: →
It states that If fourier transform of function $f(x)$ is $F(k)$ then fourier transform of function $f(x-a)$ is $F(k)e^{-iak}$, where a = constant.

Proof:- By definition fourier transform of function $f(x-a)$ is,

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ikx} f(x-a)\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ik(x-a)}\, e^{-iak} \cdot f(x-a)\, d(x-a)$$

$[\because d(x-a) = dx]$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-iky} e^{-iak} f(y)\,dy$$

where $y = x-a$

$$= \frac{e^{-iak}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-iky} f(y)\,dy$$

$$= e^{-iak} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-iky} f(y)\,dy$$

$$= F(k) e^{-iak}$$

Hence proved.

(3) Modulation theoram:→

It states that if fourier transform of f(x) is F(k), then fourier transform of function $f(x)\cos\omega x$ is $\frac{1}{2}[F(k-\omega)+F(k+\omega)]$.

Proof:→ By definition, fourier transform of $f(x)\cos\omega x$ is,

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-ikx} f(x)\cos\omega x\,dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-ikx} f(x)\left[\frac{e^{i\omega x}+e^{-i\omega x}}{2}\right]dx$$

$$=\frac{1}{2}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-i(k-\omega)x} f(x)\,dx+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-i(k+\omega)x} f(x)\,dx\right]$$

$$=\frac{1}{2}\left[F(k-\omega)+F(k+\omega)\right]$$

Hence proved.

(4). Convolution or Faltung theoram:→

The convolution of two functions f(x) and g(x) is a function denoted by $f * g(x)$ and is defined as

$$f*(x)g(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(u)g(x-u)\,du$$

Now Convolution theoram states that fourier transform of convolution of f(x) and g(x) is equal to product of fourier transform of f(x) and g(x).

Proof:→ We know that convolution of f(x) and g(x) is given by

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$$f(x) * g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(u)\, g(x-u)\, du$$

Therefore by definition, Fourier transform of this convolution

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ikx} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(u)\, g(x-u)\, du \right] dx$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(u) \left[\int_{-\infty}^{+\infty} e^{-ikx}\, g(x-u)\, dx \right] du$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(u) \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ik(x-u)}\, g(x-u) \cdot e^{-iku}\, d(x-u) \right] du$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(u) \left[\frac{e^{-iku}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ik(x-u)}\, g(x-u)\, d(x-u) \right] du$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(u)\, e^{-iku} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-iky}\, g(y)\, dy \right] du$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(u)\, e^{-iku}\, G(k)\, du$$

$$= \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(u)\, e^{-iku}\, du \right] G(k)$$

$$= F(K)\, G(K)$$

(Hence proved.)

Applications of Fourier Transform:

(I) Evaluation of integrals:

Fourier transform can be used to solve some typical integrals. e.g.

Let $I_1 = \int_0^\infty e^{-ax} \cos kx \, dx$ —— (I)

and $I_2 = \int_0^\infty e^{-ax} \sin kx \, dx$ —— (II)

Integrating eqn (I) by parts,

$$I_1 = \cos kx \int_0^\infty e^{-ax} dx - \int_0^\infty \frac{d}{dx}(\cos kx) \int e^{-ax} dx$$

$$I_1 = \left[\frac{\cos kx \, e^{-ax}}{-a}\right]_0^\infty + \frac{k}{a}\int_0^\infty -\sin kx \, e^{-ax} dx$$

$$I_1 = -\frac{1}{a}[0 - 1] - \frac{k}{a} I_2$$

$$I_1 = \frac{1}{a} - \frac{k}{a} I_2 \quad \text{—— (III)}$$

Similarly integrating eqn (II) by parts,

$$I_2 = -\frac{1}{a}\left[e^{-ax} \sin kx\right]_0^\infty + \frac{k}{a}\int_0^\infty e^{-ax} \cos kx \, dx$$

$$I_2 = 0 + \frac{k}{a} I_1 \quad \text{—— (IV)}$$

Solving (III) and (IV)

(9)

$$\boxed{I_1 = \frac{a}{a^2+k^2}} \quad \& \quad \boxed{I_2 = \frac{K}{K^2+a^2}}$$

Here $f(x) = \sqrt{\frac{\pi}{2}}\, e^{-ax}$ and thus using this $f(x)$, I_1 & I_2 are solved.

II) Solution of differential equation: →

Let us consider a differential eqn. of nth order,

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = f(x) \quad \text{—— (I)}$$

where, $a_0, a_1, a_2, \ldots$ are constant cofficients.

Eqn (I) can be rewritten as

$$h(y) = f(x)$$

where $h(y) = a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1}y}{dx^{n-1}} + a_{n-2} \frac{d^{n-2}y}{dx^{n-2}} + \cdots + a_1 \frac{dy}{dx} + a_0 y$ —— (II)

Take Fourier transform of eqn (II) on both sides

$$T[L(y)] = a_n T\left[\frac{d^n y}{dx^n}\right] + a_{n-1} T\left[\frac{d^{n-1}y}{dx^{n-1}}\right] + \cdots + a_1 T\left[\frac{dy}{dx}\right] + a_0 T(y) \quad \text{—— (III)}$$

But By definition of Fourier transform,

$$T\left[\frac{d^n y}{dx^n}\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ikx} \frac{d^n y}{dx^n}\, dx \quad \text{—— (IV)}$$

Solving it, we get

$$T\left[\frac{d^n y}{dx^n}\right] = ik \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-ikx}\, y \frac{d^{n-1}y}{dx^{n-1}}\, dx.$$

Similarly by iteration Rest terms can also be solved.

X —— X —— X