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Mobius Transformation or Bilinear Transformation

A mapping of the form
$$w = \frac{az+b}{cz+d} \quad ; \quad ad-bc \neq 0 \quad \longrightarrow (1)$$

is called bilinear transformation or Mobius transformation.

It is also called as 'Linear fractional transformation'.

By (1), $\quad cwz + wd - az - b = 0$
which is linear in both w and z,
so, it is named as Bilinear transformation.

Note $\Rightarrow$ Here, $ad-bc \neq 0$
$\therefore$ if $ad-bc = 0$, then eqn. (1) reduces to,

$$w = \frac{a(z+b/a)}{c(z+d/c)} = \frac{a}{c}$$

$$\left(\because ad-bc = 0 \Rightarrow \frac{b}{a} = \frac{d}{c} \right.$$
$$\left. \Rightarrow \frac{z+b/a}{z+d/c} = 1 \right)$$

Hence, if $ad-bc=0$, then w will be a constant function.

and, a constant f^n is not linear and thus, $ad-bc \neq 0$ is the necessary condition for eqn. ①.

* **Critical points** :→

The points, where the conformal property doesn't hold, are called critical points, i.e, the points where either

$f(z)$ is not analytic or derivative of f is 0; are the critical points.

For exp,
for the tranformation,

$$w = f(z) = \frac{az+b}{cz+d}$$

$z = -\frac{d}{c}$ and $z = \infty$ are the critical points.

Reason :→ ∵ at $z = -\frac{d}{c}$, $f(z)$ is not analytic and at $z = \infty$, derivative of $f(z)$ i.e, $\frac{dw}{dz} = 0$

$$\frac{dw}{dz} = \frac{(cz+d)(a) - (az+b)(c)}{(cz+d)^2} = \frac{ad-bc}{(cz+d)^2}$$

$$\frac{dw}{dz} = \begin{cases} \infty, & \text{If } z = -\frac{d}{c};\ c \neq 0 \\ 0, & \text{If } z = \infty \end{cases}$$

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* Fixed points or invariant points :→

The points which coincide with their transformations are called fixed points, i.e, the points for which $f(z)$ and z are same, is known as fixed point.

If $w = f(z) = \dfrac{az+b}{cz+d}$, $ad-bc \neq 0$ →①

then, for fixed points,
put $f(z) = z = w$

i.e, $z = \dfrac{az+b}{cz+d}$

$\Rightarrow cz^2 + dz - az - b = 0$

$\Rightarrow cz^2 + z(d-a) - b = 0$

so,
$$z = \frac{(a-d) \pm \sqrt{(d-a)^2 + 4bc}}{2c} \rightarrow (*)$$

If $c=0$, $d \neq 0$ then by ①,

$f(z) = w = \dfrac{az+b}{d} = \dfrac{a}{d}z + \dfrac{b}{d}$ → ②

For fixed pt, put $w = z = f(z)$

so, $z = \dfrac{a}{d} z + \dfrac{b}{d}$ (By ②)

$z\left(1 - \dfrac{a}{d}\right) = \dfrac{b}{d}$

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so, $z = \frac{b}{d-a} \longrightarrow (**)$

Let $a-d \neq 0$, then eqn. (*) has a fixed point at $z=\infty$ and (**) will ~~...~~ give another fixed point, that will be finite.

Statement of some theorems :→

1. * Show that generally there are two values of z as fixed points for which $w=z$, but there is only one if $(a-d)^2+4bc=0$.

(This can be easily viewed by eqn. (*))

2. * Prove that if there are two distinct fixed pts Q and R, then the transformation may be of the form,

$$\frac{w-Q}{w-R} = k\left(\frac{z-Q}{z-R}\right)$$

(i.e, if we get two different fixed pts, then transformation can be expressed as this form)

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③ * Show that if there is only one fixed point say Q, then the transformation may be of the form,

$$\frac{1}{w-Q} = \frac{1}{z-Q} + k$$

(It is also known as Normal form).

④ * Show that if there is only ~~only~~ one finite fixed point P and other fixed point is ∞, then the transformation is of the form

$$w-P = k(z-P)$$

* Nature of Mobius Transformations :->

① If Mobius transformation has only one fixed pt say z_0, then nature of transformation is parabolic and thus can be expressed as, (By thm ③)

$$\frac{1}{w-z_0} = \frac{1}{z-z_0} + k \quad \text{if } z_0 \neq \infty$$

If fixed pt is infinite, then it is expressible as,

$$w = z + k \quad ; \quad z_0 = \infty.$$

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② A mobius transformation with two fixed pts z_1 and z_2 can be expressed as,
(By thm ②)

$$\frac{w-z_1}{w-z_2} = k\frac{(z-z_1)}{(z-z_2)} \quad , \text{ if } z_1, z_2 \neq \infty$$

<u>Or</u> $w-z_1 = k(z-z_1)$, if $z_2 = \infty$

(i) This transformation with two diff. fixed pts is called <u>hyperbolic</u>, if $k>0$.

(ii) Elliptic, if $k = e^{i\alpha}$; $\alpha \neq 0$

(iii) Loxodromic, if $k = ae^{i\alpha}$; $a \neq 1$
$\alpha \neq 0$

Here, a and α are both real numbers and $a>0$.

Some Numericals Based on above concept i.e, Fixed point :→

<u>Que</u> ⇒ Find fixed points, normal form and nature of following Mobius transformations,

① $w = \dfrac{3z-4}{z-1}$ →①

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sol^n ⇒ For fixed point, put $w=z$

$$\Rightarrow \frac{3z-4}{z-1} = z$$

$$\Rightarrow 3z-4 = z^2-z$$

$$\Rightarrow z^2-4z+4=0$$

$$\Rightarrow z = 2, 2$$

Thus, $z=2$ is the only fixed point.
So, the transformation is parabolic.
For Normal form,

By eqn. ①,

$$wz - w = 3z-4$$

$$wz - w - 3z + 4 = 0$$

adding and subtracting $2z$ and $2w$,

$$(wz - 2w - 2z + 4) + 2z + 2w - w - 3z = 0$$

$$[w(z-2) - 2(z-2)] + w - z = 0$$

$$(w-2)(z-2) + w - 2 + 2 - z = 0$$

$$(w-2)(z-2) + (w-2) - (z-2) = 0$$

$$1 + \frac{1}{z-2} - \frac{1}{w-2} = 0$$

Or,
$$\boxed{\frac{1}{w-2} = \frac{1}{z-2} + 1}$$

which is the required normal form.

Que ⇒ Find the fixed pts, normal form and nature of B-T $w = \frac{z}{z-2}$.

Solⁿ ⇒ For fixed pt, put $w = z$

$$\Rightarrow \quad z = \frac{z}{z-2}$$

$$\Rightarrow \quad z^2 - 2z - z = 0 \quad \Rightarrow z^2 - 3z = 0$$

$$\Rightarrow \quad z = 0, 3$$

i.e, two different fixed points.

∵ fixed pts are distinct, so transformation can be expressed as,

$$\frac{w-0}{w-3} = k\left(\frac{z-0}{z-3}\right) \longrightarrow (A)$$

but, $w = \frac{z}{z-2}$ (By given)

so,

$$\frac{w-0}{w-3} = \frac{\frac{z}{z-2}}{\frac{z}{z-2} - 3}$$

$$\frac{w-0}{w-3} = \frac{z}{-2z+6} = -\frac{1}{2}\left(\frac{z-0}{z-3}\right) \rightarrow (B)$$

By (A) and (B),

$$k = -\frac{1}{2}$$

Or, $k = \frac{1}{2} e^{i\pi}$ ⇒ Transformation is Loxodromic.