V.IMP

Bessel's Inequality :- If $\{u_1, u_2, \ldots, u_n\}$ be an orthonormal subset of an Inner Product Space V(F), then Prove that

$$\sum_{i=1}^{n} |\langle u, u_i \rangle|^2 \le \|u\|^2 \quad \text{for all } u \in V$$

Proof :- since $\{u_1, u_2, \ldots, u_n\}$ be an orthonormal subset of I.P.S. V(F).

$\therefore$ By def. of orthonormal set, we have

(i) $\langle u_i, u_j \rangle = 1$ if $i = j$, $1 \le i, j \le n$

(ii) $\langle u_i, u_j \rangle = 0$ if $i \ne j$;

Let $\quad v = u - \sum_{i=1}^{n} \langle u, u_i \rangle u_i \quad$ — (1)

we have given $u \in V(F)$ and u_i vectors we the help of u & u_i Construct v

Since $u, u_i \in V$.

$\therefore v \in V$. $\quad$ { v is L.C. of u & u_i }

Now; $\|v\|^2 = \langle v, v \rangle$

$$= \langle u - \sum_{i=1}^{n} \langle u, u_i \rangle u_i , v \rangle$$

Putting value of v by (1)

$$= \langle u, v \rangle - \sum_{i=1}^{n} \langle u, u_i \rangle \langle u_i, v \rangle$$

$$= \langle u, u - \sum_{i=1}^{n} \langle u, u_i \rangle u_i \rangle$$

$$- \sum_{i=1}^{n} \langle u, u_i \rangle \langle u_i , u - \sum_{j=1}^{n} \langle u, u_j \rangle u_j \rangle \quad \text{— (2)}$$

Again Putting value of v from (1)

We know that

$$\langle u, \alpha v \rangle = \bar{\alpha} \langle u, v \rangle.$$

So; By (2) we have.

$$\langle u, u - \sum_{i=1}^{n} \langle u, u_i \rangle u_i \rangle = \langle u, u \rangle - \langle u, \sum_{i=1}^{n} \underbrace{\langle u, u_i \rangle}_{\alpha} u_i \rangle$$

$$\{ \langle u, \alpha u_i \rangle = \bar{\alpha} \langle u, u_i \rangle \}$$

$$= \langle u, u \rangle - \sum_{i=1}^{n} \overline{\langle u, u_i \rangle} \langle u, u_i \rangle \quad (3)$$

Using (3) in (2) we get;

$$\|v\|^2 = \langle u, u \rangle - \sum_{i=1}^{n} \overline{\langle u, u_i \rangle} \langle u, u_i \rangle - \sum_{i=1}^{n} \langle u, u_i \rangle \left[\langle u_i, u \rangle - \sum_{j=1}^{n} \overline{\langle u, u_j \rangle} \langle u_i, u_j \rangle \right]$$

$$= \langle u, u \rangle - \sum_{i=1}^{n} \overline{\langle u, u_i \rangle} \langle u, u_i \rangle - \sum_{i=1}^{n} \langle u, u_i \rangle \langle u_i, u \rangle + \sum_{i=1}^{n} \langle u, u_i \rangle \sum_{j=1}^{n} \overline{\langle u, u_j \rangle} \langle u_i, u_j \rangle$$

$$\Rightarrow \|v\|^2 = \|u\|^2 - \sum_{i=1}^{n} \overline{\langle u, u_i \rangle} \langle u, u_i \rangle - \sum_{i=1}^{n} \langle u, u_i \rangle \overline{\langle u, u_i \rangle} + \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \langle u, u_i \rangle \overline{\langle u, u_j \rangle} \langle u_i, u_j \rangle \right)$$

$$\left\{ \begin{array}{l} \text{By property of I.P.S.} \\ \langle u_i, u \rangle = \overline{\langle u, u_i \rangle} \end{array} \right.$$

$$= \|u\|^2 - \sum_{i=1}^{n} |\langle u, u_i \rangle|^2 - \sum_{i=1}^{n} |\langle u, u_i \rangle|^2 + \sum_{i=1}^{n} \langle u, u_i \rangle \overline{\langle u, u_i \rangle} \cdot 1$$

$$= \|u\|^2 - 2 \sum_{i=1}^{n} |\langle u, u_i \rangle|^2 + \sum_{i=1}^{n} |\langle u, u_i \rangle|^2$$

$$\|v\|^2 = \|u\|^2 - \sum_{i=1}^{n} |\langle u, u_i \rangle|^2 \quad (4)$$

Also; $\|v\|^2 \geq 0$

$$\therefore (4) \Rightarrow \|u\|^2 - \sum_{i=1}^{n} |\langle u, u_i \rangle|^2 \geq 0$$

$$\Rightarrow \|u\|^2 \geq \sum |\langle u, u_i \rangle|^2$$

$\because \sum_{j=1}^{n} \langle u, u_i \rangle \overline{\langle u, u_j \rangle} \langle u_i, u_j \rangle$ (*)

when $j = i$ $\langle u_i, u_j \rangle = 1$

$j \neq i$ $\langle u_i, u_j \rangle = 0$

$\therefore$ (*) becomes

$\langle u, u_i \rangle \overline{\langle u, u_i \rangle} \cdot 1 = |\langle u, u_i \rangle|^2$

$\boxed{z \cdot \bar{z} = |z|^2}$

$$\Rightarrow \sum_{i=1}^{n} |\langle u, u_i \rangle|^2 \le \|u\|^2 \quad \forall\, u \in V \quad (5)$$

Corr:- Equality holds in (5)

iff $\|v\|^2 = 0$ iff $\langle v, v \rangle = 0$

Then From (4)
$$\left[\begin{aligned} 0 &= \|u\|^2 - \sum_{i=1}^{n} |\langle u, u_i \rangle|^2 \\ \Rightarrow \|u\|^2 &= \sum_{i=1}^{n} |\langle u, u_i \rangle|^2 \end{aligned}\right]$$

$\langle v, v \rangle = 0$ iff $v = 0$ [By Property of I.P.S. $\langle u, u \rangle = 0$ iff $u = 0$]

$\Leftrightarrow v = 0$

$\Leftrightarrow u - \sum_{i=1}^{n} \langle u, u_i \rangle u_i = 0$ [By (1)]

$$\Leftrightarrow u = \sum_{i=1}^{n} \langle u, u_i \rangle u_i$$

$$u = \langle u, u_1 \rangle u_1 + \langle u, u_2 \rangle u_2 + \longrightarrow + \langle u, u_n \rangle u_n$$

$$u = \alpha_1 u_1 + \alpha_2 u_2 + \longrightarrow + \alpha_n u_n$$

$\Leftrightarrow u$ is a L.C. of $u_1, u_2, \longrightarrow u_n$

$\Leftrightarrow u \in L(S)$ where $S = \{u_1, u_2, \longrightarrow u_n\}$

$\Leftrightarrow \{u_1, u_2, \longrightarrow, u_n\}$ is a basis of V.