B.S.C II YEAR IV SEM

UNIT – I

FOURIER THEOREM :→

Fourier theoram was given by french mathematician JBT Fourier, according to which any periodic complex function can be considered as made up of simple harmonic waves of definite amplitude, phases and periods.
In simple words, complex waves are easily splitted up into sine and/or cosine waveforms.
If f(x) is a complex wave, then according to fourier theoram, it can be represented in form of Fourier series as:–

$$\boxed{f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx} \quad \text{——— (I)}$$

Where a_0, a_n and b_n are called fourier coefficients.

Evaluation of fourier cofficients in range $(-\pi, \pi)$:→

(i) Determination of a_0 :→

To determine a_0, integrate eq^n (I) on both sides within limits $(-\pi, \pi)$.

$$\int_{-\pi}^{+\pi} f(x)dx = \int_{-\pi}^{+\pi} a_0 dx + \int_{-\pi}^{+\pi} \sum_{n=1}^{\infty} a_n \cos nx \, dx + \int_{-\pi}^{+\pi} \sum_{n=1}^{\infty} b_n \sin nx$$

$$\Rightarrow \int_{-\pi}^{+\pi} f(x)dx = a_0 [x]_{-\pi}^{+\pi} + a_n \int_{-\pi}^{+\pi} \cos nx \, dx + b_n \int_{-\pi}^{+\pi} \sin nx \, dx$$

$$\Rightarrow \int_{-\pi}^{+\pi} f(x)dx = a_0 [\pi - (-\pi)] + \frac{a_n}{n} [\sin nx]_{-\pi}^{+\pi} + \frac{b_n}{n} [-\cos nx]_{-\pi}^{+\pi}$$

$$\Rightarrow \int_{-\pi}^{+\pi} f(x)dx = a_0 [2\pi] + \frac{a_n}{n} [\sin n\pi + \sin n\pi] + \frac{b_n}{n} [-\cos n\pi + \cos n\pi]$$

$$\int_{-\pi}^{\pi} f(x)\,dx = a_0 2\pi + \frac{a_n}{n}[0] - \frac{b_n}{n}\left[\cancel{\cos n\pi} - \cancel{\cos n\pi}\right]$$

$$\Rightarrow \int_{-\pi}^{+\pi} f(x) = a_0 2\pi$$

$$\Rightarrow \boxed{a_0 = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(x)\,dx}$$

(ii) <u>determination of a_n</u>: $\rightarrow$

To determine a_n, multiply eqn (I) on both sides by $\cos nx$, we get,

$$\int_{-\pi}^{+\pi} f(x)\cos nx\,dx = \int_{-\pi}^{+\pi} a_0 \cos nx + \int_{-\pi}^{+\pi}\sum_{n=1}^{\infty} a_n \cos^2 nx + \int_{-\pi}^{+\pi}\sum_{n=1}^{\infty} b_n \sin nx \cos nx$$

$$\Rightarrow \int_{-\pi}^{+\pi} f(x)\cos nx\,dx = a_0\left[\sin nx\right]_{-\pi}^{+\pi} + \frac{a_n}{2}\int_{-\pi}^{+\pi} 2\cos^2 nx\,dx + \frac{b_n}{2}\int_{-\pi}^{+\pi} 2\sin nx\cos nx\,dx$$

$$\Rightarrow \int_{-\pi}^{+\pi} f(x)\cos nx\,dx = a_0\left[\sin \pi - \sin(-n\pi)\right] + \frac{a_n}{2}\int_{-\pi}^{+\pi}(1+\cos 2nx)\,dx + \frac{b_n}{2}\int_{-\pi}^{+\pi}\sin 2nx\,dx$$

$[\because \sin n\pi = 0]$

$$\Rightarrow \int_{-\pi}^{+\pi} f(x)\cos nx\,dx = a_0[0] + \frac{a_n}{2}\int_{-\pi}^{+\pi} dx + \frac{a_n}{2}\int_{-\pi}^{+\pi}\cos 2nx\,dx + \left(\frac{-b_n}{2\cdot 2n}\right)\left[\cos 2nx\right]_{-\pi}^{+\pi}$$

$$\Rightarrow \int_{-\pi}^{+\pi} f(x)\cos nx\,dx = \frac{a_n}{2}\left[\pi - (-\pi)\right] + \frac{a_n}{2\cdot 2n}\left[\sin 2nx\right]_{-\pi}^{+\pi} - \frac{b_n}{4n}\left[\cos 2n\pi - \cos(-2n\pi)\right]$$

$$\Rightarrow \int_{-\pi}^{+\pi} f(x)\cos nx\,dx = \frac{a_n}{\cancel{2}}\left[\cancel{2}\pi\right] + \frac{a_n}{4n}\left[\sin 2n\pi - \sin 2n(-\pi)\right] - \frac{b_n}{4n}\left[\cancel{\cos 2n\pi} - \cancel{\cos 2n\pi}\right]$$

$$\Rightarrow \int_{-\pi}^{+\pi} f(x)\cos nx\,dx = a_n\pi + \frac{a_n}{4n}[0] - \frac{b_n}{4n}[0]$$

$$\Rightarrow \boxed{a_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\cos nx\,dx}$$

Determination of b_n :—

To determine b_n, multiply eqn (I) on both sides by $\sin nx$ and integrate within limits $(-\pi, \pi)$.

$$\int_{-\pi}^{+\pi} f(x)\sin nx\,dx = \int_{-\pi}^{+\pi} a_0 \sin nx\,dx + \int_{-\pi}^{+\pi}\sum_{n=1}^{\infty} a_n \cos nx \sin nx\,dx + \int_{-\pi}^{+\pi}\sum_{n=1}^{\infty} b_n \sin^2 nx\,dx$$

$$\Rightarrow \int_{-\pi}^{+\pi} f(x)\sin nx\,dx = a_0\left[\frac{-\cos nx}{n}\right]_{-\pi}^{+\pi} + \frac{a_n}{2}\int_{-\pi}^{+\pi} 2\sin nx\cos nx\,dx + \frac{b_n}{2}\int_{-\pi}^{+\pi} 2\sin^2 nx\,dx$$

$$\Rightarrow \int_{-\pi}^{+\pi} f(x)\sin nx\,dx = \frac{-a_0}{n}\left[\cos n\pi - \cos(-n\pi)\right] + \frac{a_n}{2}\int_{-\pi}^{+\pi} \sin 2nx\,dx + \frac{b_n}{2}\int_{-\pi}^{+\pi}(1-\cos 2nx)\,dx$$

$$\Rightarrow \int_{-\pi}^{+\pi} f(x)\sin nx\,dx = -\frac{a_0}{n}\left[\cancel{\cos n\pi} - \cancel{\cos n\pi}\right] + \frac{a_n}{2}\left[\frac{-\cos 2nx}{2n}\right]_{-\pi}^{+\pi} + \frac{b_n}{2}\int_{-\pi}^{+\pi} dx - \frac{b_n}{2}\int_{-\pi}^{+\pi}\cos 2nx\,dx$$

$$\Rightarrow \int_{-\pi}^{+\pi} f(x)\sin nx\,dx = -\frac{a_n}{4n}\left[\cos 2n\pi - \cos(-2n\pi)\right] + \frac{b_n}{2}\left[\pi - (-\pi)\right] - \frac{b_n}{2}\left[\frac{\sin 2nx}{2n}\right]_{-\pi}^{+\pi}$$

$$\Rightarrow \int_{-\pi}^{+\pi} f(x)\sin nx\,dx = \frac{-a_n}{4n}\left[\cancel{\cos 2n\pi} - \cancel{\cos 2n\pi}\right] + \frac{b_n}{\cancel{2}}\left[\cancel{2}\pi\right] - \frac{b_n}{2\cdot 2n}\left[\sin 2n\pi - \sin(-2n\pi)\right]$$

$$\Rightarrow \int_{-\pi}^{+\pi} f(x)\sin nx\,dx = b_n\pi - \frac{b_n}{4n}[0]$$

$$\Rightarrow \boxed{b_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\sin nx\,dx}$$

Importance of Fourier Theoram :— Fourier theoram finds applications in astronomy, meterology, tides and earthquakes etc. In sound, quality of musical note depends upon harmonics present in it. So analysis of Sound note by Fourier analysis enables the quality of sound to be determined.

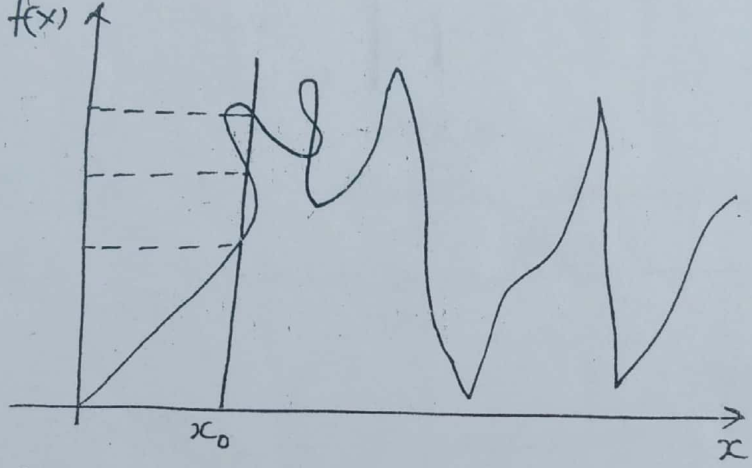
Limitations of Fourier Theorem :- → (Dirichlet Conditions)

(4)

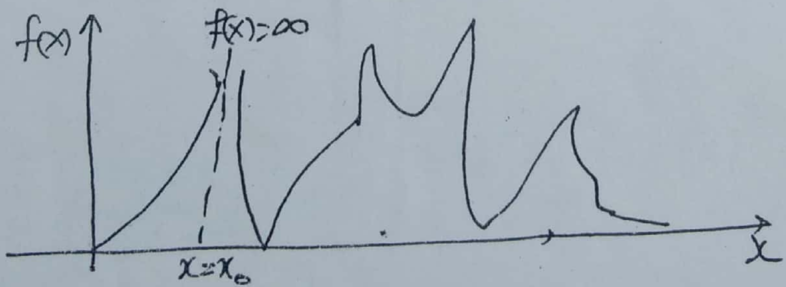
A complex function f(x) can be expanded by fourier theorem if it satisfies following conditions also called as Dirichlet conditions.

(i) The function f(x) should be single valued. It should have one value of f(x) for given value of x.

e.g function f(x) shown below cannot be expanded by Fourier theorem, because it is not single valued and it has 3 values of f(x) for a single value of x.



(ii) The complex function f(x) should have finite value. It should not have infinite value at any place. e.g function shown below can't be expanded by Fourier theorem because it has ∞ value at $x = x_0$ (het)



(iii) The function f(x) should have finite no of maxima and minima.

Rectangular wave Analysis: →

Fig 1. below shows a rectangular wave.

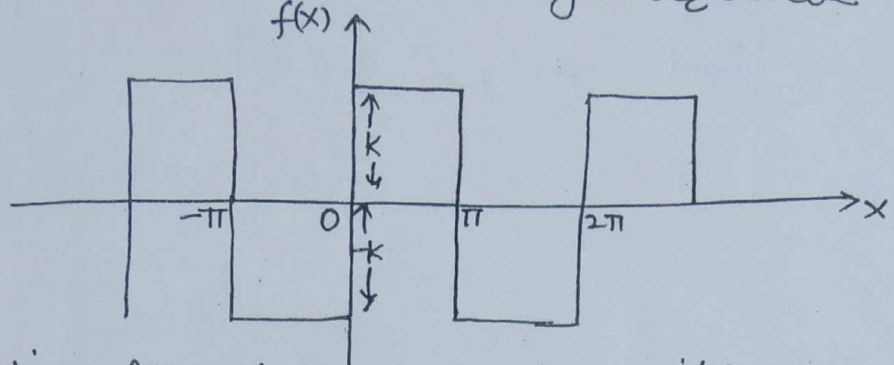


Fig 1.

The function f(x) of Fig 1 can be written as,

$$\left.\begin{aligned} f(x) &= +K \qquad 0 < x < \pi \\ &= -K \qquad -\pi < x < 0 \end{aligned}\right] \quad \text{—— (I)}$$

Fourier series for this function can be represented by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{—— (II)}$$

Where,

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(x)\,dx \;,\; a_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\cos nx\,dx \;,\; b_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\sin nx\,dx$$

determination of a_0: → $\quad a_0 = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(x)\,dx$

$$\Rightarrow a_0 = \frac{1}{2\pi}\left[\int_{-\pi}^{0} f(x)\,dx + \int_{0}^{\pi} f(x)\,dx\right]$$

$$\Rightarrow a_0 = \frac{1}{2\pi}\left[\int_{-\pi}^{0} (-K)\,dx + \int_{0}^{\pi} +K\,dx\right] = \frac{1}{2\pi}\left[(-K)[x]_{-\pi}^{0} + K[x]_{0}^{\pi}\right]$$

$$\Rightarrow a_0 = \frac{1}{2\pi}\left[(-K)[0-(-\pi)] + K[\pi - 0]\right] = \frac{1}{2\pi}\left[\cancel{-K\pi} + \cancel{K\pi}\right]$$

$$\Rightarrow \boxed{a_0 = 0}$$

determination of a_n: → $\quad a_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\cos nx\,dx$

$$\Rightarrow a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\cos nx\,dx + \int_{0}^{\pi} f(x)\cos nx\,dx\right] = \left[\frac{1}{\pi}\left(\int_{-\pi}^{0} -K\cos nx\,dx + \int_{0}^{\pi} K\cos nx\,dx\right)\right]$$

$$a_n = \frac{1}{\pi}\left[-k\int_{-\pi}^{0}\cos nx\,dx + k\int_{0}^{\pi}\cos nx\,dx\right] = \frac{K}{\pi}\left[\frac{(-1)}{n}\left[\sin nx\right]_{-\pi}^{0} + \frac{1}{n}\left[\sin nx\right]_{0}^{\pi}\right]$$

$$a_n = \frac{k}{\pi}\left[\left(\frac{-1}{n}\right)\left[\sin 0 - \sin(-n\pi)\right] + \frac{1}{n}\left[\sin n\pi - \sin 0\right]\right]$$

$$a_n = \frac{k}{\pi}\left[\left(\frac{-1}{n}\right)(0) + \frac{1}{n}(0)\right] = 0$$

$$\Rightarrow \boxed{a_n = 0}$$

Determination of b_n: $\rightarrow$ $b_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\sin nx$

$$\Rightarrow b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\sin nx\,dx + \int_{0}^{\pi} f(x)\sin nx\,dx\right]$$

$$\Rightarrow b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} -k\sin nx\,dx + \int_{0}^{\pi} k\sin nx\,dx\right] = \frac{k}{\pi}\left[(-1)\int_{-\pi}^{0}\sin nx\,dx + \int_{0}^{\pi}\sin nx\,dx\right]$$

$$\Rightarrow b_n = \frac{K}{\pi}\left[\frac{(-1)(-1)}{n}\left[\cos nx\right]_{-\pi}^{0} + \frac{(-1)}{n}\left[\cos nx\right]_{0}^{\pi}\right]$$

$$\Rightarrow b_n = \frac{K}{\pi}\left[\frac{1}{n}\left[\cos 0 - \cos(-n\pi)\right] - \frac{1}{n}\left[\cos n\pi - \cos 0\right]\right]$$

$$\Rightarrow b_n = \frac{k}{\pi}\left[\frac{1}{n}\left[1 - \cos n\pi\right] - \frac{1}{n}\left[\cos n\pi - 1\right]\right]$$

$$\Rightarrow b_n = \frac{k}{n\pi}\left[1 - \cos n\pi - \cos n\pi + 1\right] = \frac{k}{n\pi}\left[2 - 2\cos n\pi\right]$$

$$\Rightarrow b_n = \frac{2K}{n\pi}\left[1 - \cos n\pi\right] \qquad \because \left[\cos n\pi = (-1)^n\right]$$

$$\Rightarrow \boxed{b_n = \frac{2K}{n\pi}\left[1 - (-1)^n\right]}$$

Using a_0, a_n, b_n in eq^n (II), Fourier series of Rectangular wave becomes

$$f(x) = 0 + 0 + \sum_{n=1}^{\infty}\frac{2K}{n\pi}\left[1 - (-1)^n\right]\sin nx$$

$$\Rightarrow \boxed{f(x) = \frac{4K}{\pi}\sin x + \frac{4K}{3\pi}\sin 3x + \frac{4K}{5\pi}\sin 5x + \cdots}$$

Half wave Rectifier Analysis:→

Fig below shows output waveform of a half wave rectifier.

From diagram, function f(x) can be represented by,

$$\left.\begin{aligned} f(x) &= E_0 \sin x \qquad 0 < x < \pi \\ &= 0 \qquad \pi < x < 2\pi \end{aligned}\right] \quad \text{— I}$$

Here Fourier series required is between limits $(0, 2\pi)$ and not between $(-\pi, +\pi)$.
Also, Fourier series for limits $(0, 2\pi)$ is same as $(-\pi, \pi)$
So,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{—— II}$$

where

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx \;,\quad a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx \;,\quad b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx$$

determination of a_0:→ $a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx$

$$\Rightarrow a_0 = \frac{1}{2\pi}\left[\int_0^{\pi} f(x)dx + \int_{\pi}^{2\pi} f(x)\,dx\right] = \frac{1}{2\pi}\left[\int_0^{\pi} E_0 \sin x\,dx + \int_{\pi}^{2\pi} 0\,dx\right]$$

$$\Rightarrow a_0 = \frac{E_0}{2\pi}\int_0^{\pi} \sin x\,dx = \frac{-E_0}{2\pi}\left[\cos x\right]_0^{\pi} = -\frac{E_0}{2\pi}\left[\cos\pi - \cos 0\right]$$

$$\Rightarrow a_0 = \frac{-E_0}{2\pi}\left[-1-1\right] = \frac{-E_0}{\cancel{2}\pi}(-\cancel{2}) \Rightarrow \boxed{a_0 = \frac{E_0}{\pi}}$$

determination of a_n:→ $a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$

$$\Rightarrow a_n = \frac{1}{\pi}\left[\int_0^{\pi} f(x)\cos nx\,dx + \int_{\pi}^{2\pi} f(x)\cos nx\,dx\right]$$

$$a_n = \frac{1}{\pi}\left[\int_0^{\pi} E_0 \sin x \cos nx\,dx + \int_{\pi}^{2\pi} 0\cdot\cos nx\,dx\right]$$

$$\Rightarrow a_n = \frac{E_0}{\pi}\left[\frac{1}{2}\int_0^{\pi}(2\sin x\cos nx)dx\right] = \frac{E_0}{2\pi}\left[\int_0^{\pi}[\sin(n+1)x + \sin(1-n)x]dx\right]$$

$$\boxed{2\sin A\cos B = \sin(A+B) + \sin(A-B)}$$

$$\Rightarrow a_n = \frac{E_0}{2\pi}\left[\int_0^{\pi}\sin(n+1)x\,dx - \int_0^{\pi}\sin(n-1)x\,dx\right]$$

$$\Rightarrow a_n = \frac{E_0}{2\pi}\left[-\frac{1}{n+1}[\cos(n+1)x]_0^{\pi} + \frac{1}{n-1}[\cos(n-1)x]_0^{\pi}\right]$$

$$\Rightarrow a_n = \frac{E_0}{2\pi}\left[\frac{-1}{n+1}[\cos(n+1)\pi - \cos 0] + \frac{1}{n-1}[\cos(n-1)\pi - \cos 0]\right]$$

$$\Rightarrow a_n = \frac{E_0}{2\pi}\left[\frac{-1}{n+1}[\cos(n+1)\pi - 1] + \frac{1}{n-1}[\cos(n-1)\pi - 1]\right]$$

$$[\because \cos(n+1)\pi = \cos(n-1)\pi]$$

$$\Rightarrow a_n = \frac{E_0}{2\pi}\left[\frac{-1}{n+1}[\cos(n-1)\pi - 1] + \frac{1}{n-1}[\cos(n-1)\pi - 1]\right]$$

$$\Rightarrow a_n = \frac{E_0}{2\pi}\left[(\cos(n-1)\pi - 1)\left[\frac{1}{n-1} - \frac{1}{n+1}\right]\right] = \frac{E_0}{2\pi}\left[[\cos(n-1)\pi - 1]\left[\frac{\cancel{n}+1-\cancel{n}+1}{n^2-1}\right]\right]$$

$$\Rightarrow a_n = \frac{E_0}{\cancel{2}\pi}\cdot\frac{\cancel{2}}{n^2-1}[\cos(n-1)\pi - 1]$$

$$[\because \cos(n-1)\pi = (-1)^{n-1}]$$

$$\Rightarrow \boxed{a_n = \frac{E_0}{(n^2-1)\pi}\left[(-1)^{n-1} - 1\right]}$$

determination of b_n: $\rightarrow$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$\Rightarrow b_n = \frac{1}{\pi}\left[\int_0^{\pi} f(x)\sin nx\,dx + \int_0^{2\pi} f(x)\sin nx\,dx\right]$$

$$b_n = \frac{1}{\pi}\left[\int_0^{\pi} E_0 \sin x \sin nx\,dx + \int_{\pi}^{2\pi} 0\cdot \sin nx\,dx\right] = \frac{E_0}{\pi}\int_0^{\pi} \sin x \sin nx\,dx \quad \text{--- (III)}$$

$$b_n = \frac{E_0}{2\pi}\int_0^{\pi} 2\sin x \sin nx\,dx = \frac{E_0}{2\pi}\left[\int_0^{\pi}\left[\cos(1-n)x - \cos(n+1)x\right]dx\right]$$

$$\boxed{2\sin A \sin B = \cos(A-B) - \cos(A+B)}$$

$$b_n = \frac{E_0}{2\pi}\left[\int_0^{\pi}\cos(n-1)x\,dx - \int_0^{\pi}\cos(n+1)x\,dx\right] = \frac{E_0}{2\pi}\left[\frac{1}{n-1}\left[\sin(n-1)x\right]_0^{\pi} - \frac{1}{n+1}\left[\sin(n+1)x\right]_0^{\pi}\right]$$

$$b_n = \frac{E_0}{2\pi}\left[\frac{1}{n-1}\left[\sin(n-1)\pi - \sin 0\right] - \frac{1}{n+1}\left[\sin(n+1)\pi - \sin 0\right]\right] \qquad [\because \sin n\pi = 0]$$

$$b_n = \frac{E_0}{2\pi}[0] = 0$$

$$\Rightarrow \boxed{b_n = 0}$$

If we put $n=1$ in eqn (III), we get b_1 as follow.

$$b_1 = \frac{E_0}{\pi}\int_0^{\pi}\sin x \sin x\,dx = \frac{E_0}{2\pi}\int_0^{\pi} 2\sin^2 x\,dx = \frac{E_0}{2\pi}\int_0^{\pi}(1-\cos 2x)\,dx$$

$$b_1 = \frac{E_0}{2\pi}\int_0^{\pi}dx - \frac{E_0}{2\pi}\int_0^{\pi}\cos 2x\,dx = \frac{E_0}{2\pi}[x]_0^{\pi} - \frac{E_0}{2\pi\times 2}[\sin 2x]_0^{\pi}$$

$$b_1 = \frac{E_0}{2\pi}[\pi - 0] - \frac{E_0}{4\pi}[\sin 2\pi - \sin 0] = \frac{E_0}{2\cancel{\pi}}\times\cancel{\pi} - 0$$

$b_1 = \frac{E_0}{2} \neq 0$. b_1 is non zero, however for $n>1$, $b_n = 0$

So using a_0, a_n and b_n in eqn (II), we get,

$$f(x) = \frac{E_0}{\pi} + \sum_{n=1}^{\infty}\frac{E_0}{(n^2-1)\pi}\left[(-1)^{n+1}-1\right]\cos nx + \frac{E_0}{2}\sin x$$

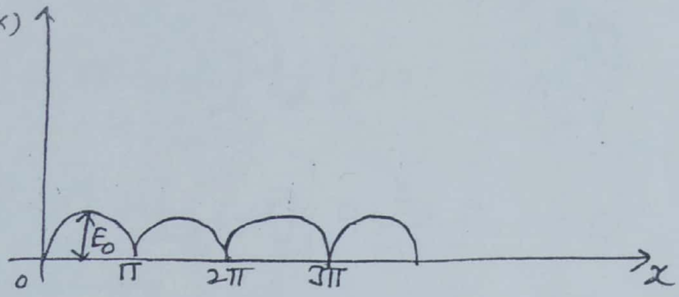
$$\Rightarrow f(x) = \frac{E_0}{\pi} + \frac{-2E_0}{3\pi}\cos 2x + \frac{-2E_0}{15\pi}\cos 4x + \cdots + \frac{E_0}{2}\sin x$$

$$\Rightarrow \boxed{f(x) = \frac{E_0}{\pi} - \frac{2E_0}{3\pi}\cos 2x - \frac{2E_0}{15\pi}\cos 4x + \cdots + \frac{E_0}{2}\sin x}$$

$$\text{Ripple factor} = \sqrt{\left(\frac{E_{rms}^2}{E_{dc}^2} - 1\right)} = \sqrt{\frac{(E_0/2)^2}{(E_0/\pi)^2} - 1} = 1.21$$

Full wave Rectifier: →

Fig below shows waveform of a full wave rectifier.



The function f(x) for it can be represented mathematically as

$$f(x) = \begin{cases} E_0 \sin x & 0 < x < \pi \\ = -E_0 \sin x & \pi < x < 2\pi \end{cases} \quad \text{—— (I)}$$

Fourier series in interval $(0, 2\pi)$ is expressed as,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{—— (II)}$$

determination of a_0: →

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{2\pi}\left[\int_0^{\pi} f(x)\,dx + \int_{\pi}^{2\pi} f(x)\,dx\right]$$

$$a_0 = \frac{1}{2\pi}\left[\int_0^{\pi} E_0 \sin x\,dx + \int_{\pi}^{2\pi} -E_0 \sin x\,dx\right]$$

$$a_0 = \frac{E_0}{2\pi}\left[(-1)\left[\cos x\right]_0^{\pi} + \left[\cos x\right]_{\pi}^{2\pi}\right] = \frac{E_0}{2\pi}\left[(-1)\left[\cos\pi - \cos 0\right] + \left[\cos 2\pi - \cos\pi\right]\right]$$

$$a_0 = \frac{E_0}{2\pi}\left[(-1)\left[-1-1\right] + \left[1+1\right]\right] = \frac{E_0}{2\pi} \times 4 2 = \frac{2E_0}{\pi}$$

$$\boxed{a_0 = \frac{2E_0}{\pi}}$$

determination of a_n: → $a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$

$$\Rightarrow a_n = \frac{1}{\pi}\left[\int_0^{\pi} f(x)\cos nx\,dx + \int_{\pi}^{2\pi} f(x)\cos nx\,dx\right] = \frac{1}{\pi}\left[\int_0^{\pi} E_0 \sin x \cos nx\,dx + \int_{\pi}^{2\pi} -E_0 \sin x \cos nx\,dx\right]$$

$$\Rightarrow a_n = \frac{E_0}{\pi}\left[\int_0^{\pi} \sin x \cos nx\,dx - \int_{\pi}^{2\pi} \sin x \cos nx\,dx\right] \quad \text{—— (III)}$$

Let us find a_n at $n = 1$,

$$a_1 = \frac{E_0}{\pi}\left[\int_0^{\pi} \sin x \cos x\,dx - \int_{\pi}^{2\pi} \sin x \cos x\,dx\right] = \frac{E_0}{2\pi}\left[\int_0^{\pi} 2\sin x \cos x\,dx - \int_{\pi}^{2\pi} 2\sin x \cos x\right)$$

$$a_1 = \frac{E_0}{2\pi}\left[\int_0^{\pi} \sin 2x\,dx - \int_{\pi}^{2\pi} \sin 2x\,dx\right] = \frac{E_0}{2\pi}\left[-\left[\frac{\cos 2x}{2}\right]_0^{\pi} + \left[\frac{\cos 2x}{2}\right]_{\pi}^{2\pi}\right]$$

$$a_1 = \frac{E_0}{2\pi}\left[-\frac{1}{2}\left[\cos \pi - \cos 0\right] + \frac{1}{2}\left[\cos 2\pi - \cos\pi\right]\right]$$

$$a_1 = \frac{E_0}{2\pi}\left[-\frac{1}{2}\left[\cancel{1}-\cancel{1}\right] + \frac{1}{2}\left[\cancel{1}+\cancel{1}\right]\right] = \frac{E_0}{2\pi}[0] = 0$$

$$\boxed{a_1 = 0}$$

Now for $n > 1$,

a_n from eqn (III) can be derived as:-

$$a_n = \frac{E_0}{\pi}\left[\int_0^{\pi} \sin x \cos nx\,dx - \int_{\pi}^{2\pi} \sin x \cos nx\,dx\right]$$

$$a_n = \frac{E_0}{2\pi}\left[\int_0^{\pi} 2\sin x \cos nx\,dx - \int_{\pi}^{2\pi} 2\sin x \cos nx\,dx\right]$$

$$a_n = \frac{E_0}{2\pi}\left[\int_0^{\pi}[\sin(n+1)x\,dx - \sin(n-1)x\,dx] - \int_{\pi}^{2\pi}(\sin(n+1)x\,dx - \sin(n-1)x\,dx)\right]$$

$$a_n = \frac{E_0}{2\pi}\left[-\frac{1}{n+1}\left[\cos(n+1)x\right]_0^{\pi} + \frac{1}{n-1}\left[\cos(n-1)x\right]_0^{\pi} + \frac{1}{n+1}\left(\cos(n+1)x\right)_{\pi}^{2\pi} - \frac{1}{n-1}\left[\cos(n-1)x\right]_{\pi}^{2\pi}\right]$$

$$a_n = \frac{E_0}{2\pi}\left[-\frac{1}{n+1}\left[\cos(n+1)\pi - \cos 0\right] + \frac{1}{n-1}\left[\cos(n-1)\pi - \cos 0\right] + \frac{1}{n+1}\left[\cos(n+1)2\pi - \cos(n+1)\pi\right] - \frac{1}{n-1}\left[\cos(n-1)2\pi - \cos(n-1)\pi\right]\right]$$

$[\because \cos(n+1)\pi = \cos(n-1)\pi]$

$$a_n = \frac{E_0}{2\pi}\left[\frac{-1}{n+1}\left[\cos(n-1)\pi - 1\right] + \frac{1}{n-1}\left[\cos(n-1)\pi - 1\right] + \frac{1}{n+1}\left[1 - \cos(n-1)\pi\right] - \frac{1}{n-1}\left[1 - \cos(n-1)\pi\right]\right]$$

$[\because \cos(n+1)2\pi = \cos(n-1)2\pi = 1]$

$$a_n = \frac{E_0}{2\pi}\left[\left[\cos(n-1)\pi - 1\right]\left[-\frac{1}{n+1} + \frac{1}{n-1} - \frac{1}{n+1} + \frac{1}{n-1}\right]\right] = \frac{E_0}{2\pi}\left(\cos(n-1)\pi - 1\right)\left[\frac{2}{n-1} - \frac{2}{n+1}\right]$$

$\cos n\pi = (-1)^n$

$$a_n = \frac{E_0}{2\pi} \times 2\left[\frac{1}{n-1} - \frac{1}{n+1}\right]\left[\cos(n-1)\pi - 1\right] = \frac{E_0}{\pi}\left[\frac{\cancel{n}+1-\cancel{n}+1}{n^2-1}\right]\left[\cos(n-1)\pi - 1\right]$$

$$a_n = \frac{2E_0}{\pi(n^2-1)}\left[(-1)^{n-1} - 1\right]$$

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determination of b_n: $\rightarrow$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_0^{\pi} f(x)\sin nx\,dx + \int_{\pi}^{2\pi} f(x)\sin nx\,dx\right]$$

$$b_n = \frac{1}{\pi}\left[\int_0^{\pi} E_0\sin x\sin nx\,dx + \int_{\pi}^{2\pi} -E_0\sin x\sin nx\,dx\right]$$

$$b_n = \frac{E_0}{\pi}\left[\frac{1}{2}\int_0^{\pi} 2\sin x\sin nx\,dx - \frac{1}{2}\int_{\pi}^{2\pi} 2\sin x\sin nx\,dx\right] \quad \text{—— IV}$$

$$b_n = \frac{E_0}{2\pi}\left[\int_0^{\pi}[\cos(n-1)x - \cos(n+1)x]\,dx - \int_{\pi}^{2\pi}[\cos(n-1)x - \cos(n+1)x]\,dx\right]$$

$$b_n = \frac{E_0}{2\pi}\left[\frac{1}{n-1}[\sin(n-1)x]_0^{\pi} - \frac{1}{n+1}[\sin(n+1)x]_0^{\pi} - \frac{1}{n-1}(\sin(n-1)x]_{\pi}^{2\pi} + \frac{1}{n+1}(\sin(n+1)x]_{\pi}^{2\pi}\right]$$

$$b_n = \frac{E_0}{2\pi}\left[\frac{1}{n-1}[\sin(n-1)\pi - \sin 0] - \frac{1}{n+1}[\sin(n+1)\pi - \sin 0] - \frac{1}{n-1}[\sin(n-1)2\pi - \sin(n-1)\pi] + \frac{1}{n+1}[\sin(n+1)2\pi - \sin(n+1)\pi]\right]$$

$$\boxed{b_n = 0}$$

Let us check for $n = 1$ from eq^n IV above,

$$b_1 = \frac{E_0}{\pi}\left[\frac{1}{2}\int_0^{\pi} 2\sin x\sin x\,dx - \frac{1}{2}\int_{\pi}^{2\pi} 2\sin x\sin x\,dx\right]$$

$$b_1 = \frac{E_0}{2\pi}\left[\int_0^{\pi} 2\sin^2 x\,dx - \int_{\pi}^{2\pi} 2\sin^2 x\,dx\right] = \frac{E_0}{2\pi}\left[\int_0^{\pi}(1-\cos 2x)\,dx - \int_{\pi}^{2\pi}(1-\cos 2x)\,dx\right]$$

$$b_1 = \frac{E_0}{2\pi}\left[\int_0^{\pi} dx - \int_0^{\pi}\cos 2x\,dx - \int_{\pi}^{2\pi} dx + \int_{\pi}^{2\pi}\cos 2x\,dx\right]$$

$$b_1 = \frac{E_0}{2\pi}\left[(x]_0^{\pi} - \left[\frac{\sin 2x}{2}\right]_0^{\pi} - (x]_{\pi}^{2\pi} + \left[\frac{\sin 2x}{2}\right]_{\pi}^{2\pi}\right]$$

$$b_1 = \frac{E_0}{2\pi}\left[(\pi - 0) - \frac{1}{2}(\sin 2\pi - \sin 0) - [2\pi - \pi] + \frac{1}{2}(\sin 4\pi - \sin 2\pi)\right]$$

$$b_1 = \frac{E_0}{2\pi}\left[\cancel{\pi} - \frac{1}{2}(0) - \cancel{\pi} + \frac{1}{2}(0)\right]$$

$$\boxed{b_1 = 0}$$

Using these values in eq II, Fourier series for full wave rectifier becomes,

$$f(x) = \frac{2E_0}{\pi} + \sum_{n=1}^{\infty}\frac{2E_0}{\pi(n^2-1)}\left[(-1)^{n+1} - 1\right]\cos nx + 0$$

$$f(x) = \frac{2E_0}{\pi} + \frac{2E_0}{\pi \times 3}\left[-2\right]\cos 2x + \frac{2E_0}{\pi \times 15}(-2)\cos 4x + \text{------}$$

$$\boxed{f(x) = \frac{2E_0}{\pi} - \frac{4E_0}{3\pi}\cos 2x - \frac{4E_0}{15\pi}\cos 4x + \text{------}}$$

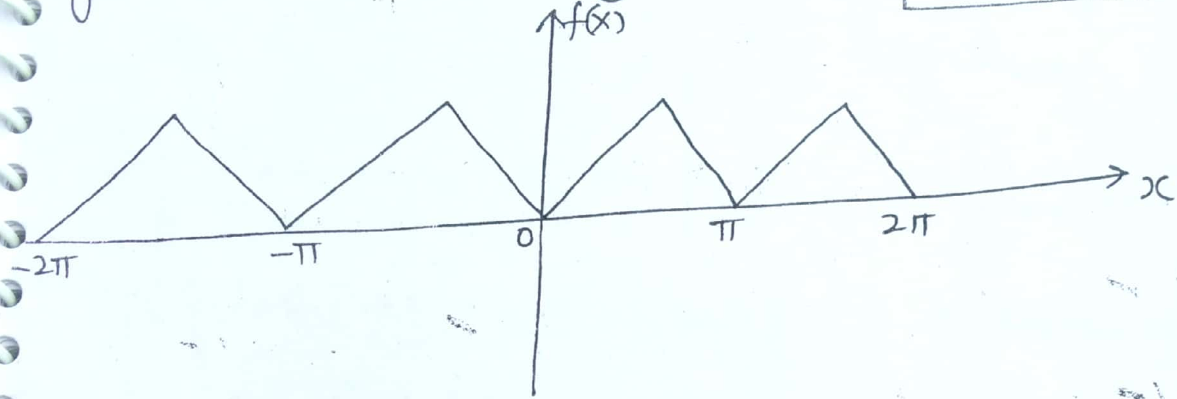
Ripple factor

$$= \sqrt{\frac{E_{rms}^2}{E_{dc}^2} - 1} =$$

$$= \sqrt{\frac{(E_0/\sqrt{2})^2}{(2E_0/\pi)^2} - 1} = 0.48$$

Triangular wave Analysis: →

Fig below shows a triangular wave.



Mathematically f(x) can be represented as

$$\left.\begin{aligned} f(x) &= Kx & 0 < x < \pi \\ &= -Kx & -\pi < x < 0 \end{aligned}\right] \text{ — (I)}$$

[∵ eqn of line is $y = mx$ and $m = K$
K is a constant]

Fourier series is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \text{ — (II)}$$

Determination of a_0: →

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(x)\,dx = \frac{1}{2\pi}\left[\int_{-\pi}^{0} f(x)\,dx + \int_{0}^{\pi} f(x)\,dx\right]$$

$$a_0 = \frac{1}{2\pi}\left[\int_{-\pi}^{0} -Kx\,dx + \int_{0}^{\pi} Kx\,dx\right] = \frac{K}{2\pi}\left[(-1)\int_{-\pi}^{0} x\,dx + \int_{0}^{\pi} x\,dx\right]$$

$$a_0 = \frac{K}{2\pi}\left[(-1)\left[\frac{x^2}{2}\right]_{-\pi}^{0} + \left\{\frac{x^2}{2}\right\}_{0}^{\pi}\right] = \frac{K}{2\pi}\left\{-\frac{1}{2}\left[0^2 - (-\pi)^2\right] + \frac{1}{2}\left[\pi^2 - 0^2\right]\right\}$$

$$a_0 = \frac{K}{2\pi \cdot 2}\left[(-1)\left[-\pi^2\right] + \pi^2\right] = \frac{K}{\cancel{4\pi}_{2}} \times \cancel{2}\pi^{\cancel{2}} = \frac{K\pi}{2}$$

Determination of a_n: →

$$a_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\cos nx\,dx + \int_{0}^{\pi} f(x)\cos nx\,dx\right]$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} -Kx\cos nx\,dx + \int_{0}^{\pi} Kx\cos nx\,dx\right]$$

$$a_n = \frac{K}{\pi}\left[(-1)\int_{-\pi}^{0} x\cos nx\,dx + \int_{0}^{\pi} x\cos nx\,dx\right] = \frac{K}{\pi}\left[\frac{(-1)}{n}\left[x\sin nx\right]_{-\pi}^{0} + \int_{-\pi}^{0}\frac{d}{dx}(x)\int\cos nx\,dx + \frac{1}{n}\left[x\sin nx\right]_{0}^{\pi} - \int_{0}^{\pi}\frac{d}{dx}(x)\int\cos nx\,dx\right]$$

$$a_n = \frac{K}{\pi}\left[-\frac{1}{n}\left[0-(-\pi)\sin(-n\pi)\right] + \frac{1}{n}\int_{-\pi}^{0}\sin nx\,dx + \frac{1}{n}\left[\pi\sin n\pi - 0\right] - \frac{1}{n}\int_{0}^{\pi}\sin nx\,dx\right]$$

$$a_n = \frac{K}{\pi}\left[\cancel{-\frac{1}{n}[0]} - \frac{1}{n^2}\left[\cos nx\right]_{-\pi}^{0} + \cancel{\frac{1}{n}[0]} + \frac{1}{n^2}\left[\cos nx\right]_{0}^{\pi}\right] = \frac{K}{\pi}\left[-\frac{1}{n^2}\left[\cos 0 - \cos(-n\pi)\right] + \frac{1}{n^2}\left[\cos n\pi - \cos 0\right]\right]$$

$$a_n = \frac{K}{\pi n^2}\left[-1 + (-1)^n + (-1)^n - 1\right] = \frac{2K}{n^2\pi}\left[(-1)^n - 1\right]$$

$$\boxed{a_n = -\frac{2K}{n^2\pi}\left[1-(-1)^n\right]}$$

Determination of b_n: →

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{+\pi} f(x)\sin nx\,dx\right] = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\sin nx\,dx + \int_{0}^{\pi} f(x)\sin nx\,dx\right]$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} -Kx\sin nx\,dx + \int_{0}^{\pi} Kx\sin nx\,dx\right] = \frac{K}{\pi}\left[\int_{0}^{\pi} x\sin nx\,dx - \int_{-\pi}^{0} x\sin nx\,dx\right]$$

$$b_n = \frac{K}{\pi}\left[x\int_{0}^{\pi}\sin nx\,dx - \int_{0}^{\pi}\frac{d}{dx}(x)\int\sin nx\,dx - x\int_{-\pi}^{0}\sin nx\,dx + \int_{-\pi}^{0}\frac{d}{dx}(x)\int\sin nx\,dx\right]$$

$$b_n = \frac{K}{\pi}\left[-\frac{1}{n}\left[x\cos nx\right]_{0}^{\pi} + \frac{1}{n}\int_{0}^{\pi}\cos nx\,dx + \frac{1}{n}\left[x\cos nx\right]_{-\pi}^{0} - \frac{1}{n}\int_{-\pi}^{0}\cos nx\,dx\right]$$

$$b_n = \frac{K}{\pi}\left[-\frac{1}{n}\left[\pi\cos n\pi - 0\right] + \frac{1}{n^2}\left[\sin nx\right]_{0}^{\pi} + \frac{1}{n}\left[0 - (-\pi)\cos(-n\pi)\right] - \frac{1}{n^2}\left[\sin nx\right]_{-\pi}^{0}\right]$$

$$b_n = \frac{K}{\pi}\left[-\cancel{\frac{1}{n}(\pi\cos n\pi)} + \frac{1}{n^2}\left[\sin n\pi - \sin 0\right] + \cancel{\frac{1}{n}\pi\cos n\pi} - \frac{1}{n^2}\left[\sin 0 - \sin(-n\pi)\right]\right]$$

$$b_n = \frac{K}{\pi}\left[\frac{1}{n^2}[0] - \frac{1}{n}[0]\right] = 0$$

$$\boxed{b_n = 0}$$

Using a_0, a_n and b_n in eq^n (II), we get fourier series for Angular wave:-

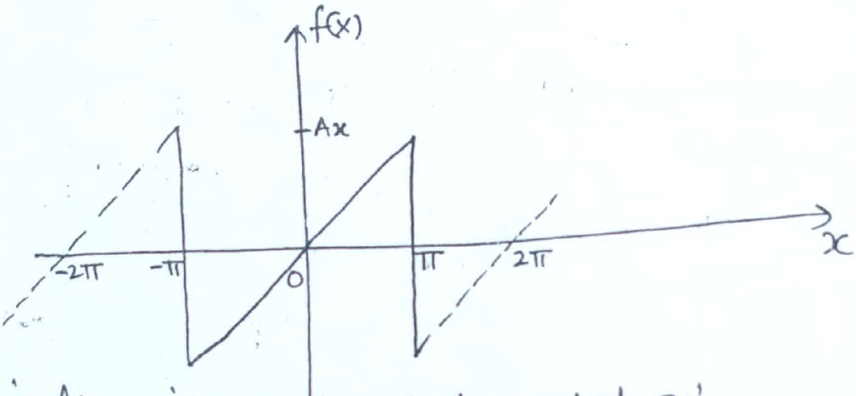
$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$f(x) = \frac{K\pi}{2} + \sum_{n=1}^{\infty} \frac{-2K}{n^2\pi}\left[1-(-1)^n\right]^{\cos nx} + 0$$

$$\Rightarrow \boxed{f(x) = \frac{K\pi}{2} - \frac{4K}{\pi}\cos x - \frac{4K}{9\pi}\cos 3x + \cdots}$$

Saw Tooth wave Analysis:→

Fig 1. below shows a saw tooth wave.



Mathematically, it can be represented as:-

$$f(x) = Ax \qquad -\pi < x < \pi \quad \text{——— (I)}$$

Fourier series in limits $(-\pi, \pi)$ can be written as,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{——— (II)}$$

determination of a_0:→

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(x)dx = \frac{1}{2\pi}\int_{-\pi}^{+\pi} Ax\,dx = \frac{A}{2\pi}\int_{-\pi} x\,dx = \frac{A}{2\pi}\left[\frac{x^2}{2}\right]_{-\pi}^{+\pi}$$

$$a_0 = \frac{A}{2\pi\cdot 2}\left[\pi^2 - (-\pi)^2\right] = \frac{A}{4\pi}\left[\cancel{\pi^2} - \cancel{\pi^2}\right] = 0$$

$$\boxed{a_0 = 0}$$

determination of a_n:→

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{+\pi} f(x)\cos nx\,dx\right] = \frac{1}{\pi}\left[\int_{-\pi}^{+\pi} Ax\cos nx\,dx\right] = \frac{A}{\pi}\int_{-\pi}^{+\pi} x\cos nx\,dx$$

$$\Rightarrow a_n = \frac{A}{\pi}\left[x\int_{-\pi}^{+\pi}\cos nx\,dx - \int_{-\pi}^{+\pi}\frac{d}{dx}(x)\int\cos nx\,dx\right]$$

$$\Rightarrow a_n = \frac{A}{\pi}\left[\frac{1}{n}\left[x\sin nx\right]^{+\pi} - \frac{1}{n}\int^{+\pi}\sin nx\,dx\right] = \frac{A}{n\pi}\left[\pi\sin n\pi - (-\pi)\sin(-n\pi) + \frac{1}{n^2}\left[\cos nx\right]^{+\pi}_{-}\right]$$

$$a_n = \frac{A}{n\pi}\left[0 + \frac{1}{n^2}\left[\cancel{\cos n\pi} - \cancel{\cos(-n\pi)}\right]\right]$$

$$a_n = \frac{A}{n\pi}[0] = 0$$

$$\Rightarrow \boxed{a_n = 0}$$

determination of b_n: $\rightarrow$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{+\pi} Ax\sin nx\,dx = \frac{A}{\pi}\int_{-\pi}^{+\pi} x\sin nx\,dx$$

$$b_n = \frac{A}{\pi}\left[x\int_{-\pi}^{+\pi}\sin nx\,dx - \int_{-\pi}^{+\pi}\frac{d}{dx}(x)\int\sin nx\,dx\right]$$

$$b_n = \frac{A}{\pi}\left[-\frac{1}{n}\left[x\cos nx\right]_{-\pi}^{+\pi} + \frac{1}{n}\int_{-\pi}^{+\pi}\cos nx\,dx\right] = \frac{A}{n\pi}\left[(-1)\left[\pi\cos n\pi - (-\pi)\cos(-n\pi)\right] + \frac{1}{n}\left[\sin nx\right]_{-\pi}^{+\pi}\right]$$

$$b_n = \frac{A}{n\cancel{\pi}}\left[(-1)\left[2\cancel{\pi}\cos n\pi\right] + \frac{1}{n}[0]\right] = -\frac{2A}{n}\cos n\pi$$

$$\boxed{b_n = -\frac{2A}{n}(-1)^n}$$

Using, a_0, a_n, b_n in eqn (II), fourier series for saw tooth wave becomes,

$$f(x) = 0 + 0 + \sum_{n=1}^{\infty}\frac{-2A}{n}(-1)^n\sin nx$$

$$f(x) = -\frac{2A}{1}(-1)^1\sin x + \left(\frac{-2A}{2}\right)(-1)^2\sin 2x + \left(\frac{-2A}{3}\right)(-1)^3\sin 3x + \text{----}$$

$$\boxed{f(x) = 2A\sin x - \frac{2A}{2}\sin 2x + \frac{2A}{3}\sin 3x + \text{----}}$$

Even function and Fourier series: $\rightarrow$

If $f(-x) = f(x)$, then $f(x)$ is called even function.

e.g $f(x) = x^2$, $f(x) = \cos x$ etc. are even functions. as we can see, when we change x by $-x$, $f(x)$ remains unchanged.

$$f(x) = x^2$$

$$f(-x) = (-x)^2$$

$$= x^2$$

$$= f(x)$$

$$\Rightarrow \boxed{f(-x) = f(x)}$$

Since f(x) is an even function, so by property of even function,

$$\int_{-\pi}^{+\pi} f(x)\,dx = 2\int_{0}^{\pi} f(x)\,dx$$

$$\therefore a_0 = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(x)dx = \frac{1}{\cancel{2}\pi}\cdot\cancel{2}\int_{0}^{\pi} f(x)dx = \frac{1}{\pi}\int_{0}^{\pi} f(x)dx$$

$$\Rightarrow \boxed{a_0 = \frac{1}{\pi}\int_{0}^{\pi} f(x)dx} \quad \text{—— (I)}$$

Also, f(x) is even, cos nx by default is an even function and product of two even function f(x) cos nx is even, Hence

$$\int_{-\pi}^{+\pi} f(x)\cos nx\,dx = 2\int_{0}^{\pi} f(x)\cos nx\,dx$$

$$\therefore a_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\cdot 2\int_{0}^{\pi} f(x)\cos nx\,dx$$

$$\boxed{a_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx} \quad \text{—— (II)}$$

Also, As f(x) sin nx will be odd, so by property of odd function,

$$\int_{-\pi}^{+\pi} f(x)\sin nx\,dx = 0$$

$$\therefore b_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\sin nx\,dx = \frac{1}{\pi}(0) = 0$$

$$\Rightarrow \boxed{b_n = 0}$$

So Fourier series for an even function becomes,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + 0$$

$$\Rightarrow \boxed{f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx}$$

Where a_0 a_n are given by ...

odd function and Fourier series:—

If $f(-x) = -f(x)$, then function $f(x)$ is called odd function.

eg $f(x) = \sin x$, $f(x) = x^3$ etc. are odd functions. as we can see if we change x by $-x$,

$$f(x) = \sin x$$
$$f(-x) = \sin(-x)$$
$$= -\sin x$$
$$= -f(x)$$
$$\Rightarrow \boxed{f(-x) = -f(x)}$$

Then $f(x)$ is called odd function.

Now, since $f(x)$ is odd function of x, by property of odd function

$$\int_{-\pi}^{+\pi} f(x)\,dx = 0$$

$$\therefore a_0 = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(x)\,dx = \frac{1}{2\pi}(0) = 0$$

$$\Rightarrow \boxed{a_0 = 0}$$

Also, $f(x)\cos nx$ is odd, Hence by property of odd function,

$$\int_{-\pi}^{+\pi} f(x)\cos nx\,dx = 0$$

$$\therefore a_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\cos nx\,dx = \frac{1}{\pi}(0) = 0$$

$$\Rightarrow \boxed{a_n = 0}$$

Also, $f(x)\sin nx$ is even function, so by property of even function,

$$\int_{-\pi}^{+\pi} f(x)\sin nx\,dx = 2\int_{0}^{\pi} f(x)\sin nx\,dx$$

$$\therefore b_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\cdot 2\int_{0}^{\pi} f(x)\sin nx\,dx$$

$$\boxed{b_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx} \text{ —— } (I)$$

$\therefore$ Fourier series for odd function becomes

$\boxed{f(x) = \sum^{\infty} b_n \sin nx}$, where b_n is given by eq^n (I)

Complex form of Fourier series:-

Fourier series for a function f(x) in interval $(-\pi, +\pi)$ and satifying Dirichlet conditions is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{—— (I)}$$

where

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(x)\,dx \quad \text{—— (II)}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\cos nx\,dx \quad \text{—— (III)}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\sin nx\,dx \quad \text{—— (IV)}$$

But we know that,

$$\cos nx = \frac{e^{inx}+e^{-inx}}{2}, \quad \sin nx = \frac{e^{inx}-e^{-inx}}{2i} \quad \text{—— (V)}$$

From (I) and (V)

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\left[\frac{e^{inx}+e^{-inx}}{2}\right] + \sum_{n=1}^{\infty} b_n\left[\frac{e^{inx}-e^{-inx}}{2i}\right]$$

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[\frac{a_n - ib_n}{2}e^{inx} + \frac{a_n+ib_n}{2}e^{-inx}\right] \quad \text{—— (VI)}$$

Let $a_0 = c_0$ —— (VII)

$$\frac{a_n - ib_n}{2} = c_n \quad \text{—— (VIII)}$$

$$\frac{a_n + ib_n}{2} = c_{-n} \quad \text{—— (IX)}$$

$\therefore$ eq^n (VI) becomes,

$$f(x) = c_0 + \sum_{n=1}^{\infty}\left[c_n e^{inx} + c_{-n}e^{-inx}\right]$$

$$\Rightarrow \boxed{f(x) = \sum_{n=-\infty}^{+\infty} c_n e^{inx}}$$

This is complex form of Fourier series.

Now we will evaluate complex Fourier cofficients c_0, c_n, c_{-n}.
For this, let us consider the integral,

$$\frac{1}{2\pi}\int_{-\pi}^{+\pi} f(x)\, e^{inx} dx = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(x)\left[\cos nx + i\sin nx\right] dx$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{+\pi} f(x)\cos nx\, dx + i\int_{-\pi}^{+\pi} f(x)\sin nx\, dx\right]$$

$$= \frac{1}{2}\left(\frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\cos nx\, dx + i\frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\sin nx\, dx\right]$$

$$= \frac{1}{2}\left[a_n + i\, b_n\right] \qquad \text{[from eq}^n\text{ (III) \& (IV)]}$$

$$= c_{-n} \qquad \text{[from eq}^n\text{ (IX)]}$$

$$\therefore \boxed{c_{-n} = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(x)\, e^{inx} dx}$$

Also, consider the integral

$$\frac{1}{2\pi}\int_{-\pi}^{+\pi} f(x)\, e^{-inx} dx = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(x)\left[\cos nx - i\sin nx\right] dx$$

$$= \frac{1}{2}\left[\frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\cos nx\, dx - i\frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\sin nx\, dx\right]$$

$$= \frac{1}{2}\left[a_n - i\, b_n\right]$$

$$= c_n$$

$$\Rightarrow \boxed{c_n = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(x)\, e^{-inx} dx}$$

Fourier series in interval $(0, \pi)$:→

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

Proof:→ Please Try yourself on same lines as first article in which we derived Fourier series & a_0, a_n, b_n in limits $(-\pi, \pi)$.

Fourier series in interval $(-L, L)$, where L is any real no:→

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Proof:- Please Try yourself.

Parseval identity for Fourier series:→

Parseval's identity give a relation between average of square of function f(x) and cofficients in Fourier series.

It is given by,

$$\frac{1}{2\pi}\int_0^{2\pi} \{f(x)\}^2 dx = A_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(A_n^2 + B_n^2)$$

Proof:→ Let Fourier series for function f(x) is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{——— (I)}$$

Multiply eqn (I) on both sides by f(x) and integrate between limits $(0, 2\pi)$.

$$\int_0^{2\pi}\{f(x)\}^2 dx = a_0\int_0^{2\pi} f(x)\,dx + \sum_{n=1}^{\infty}\left[a_n\int_0^{2\pi} f(x)\cos nx\,dx + b_n\int_0^{2\pi} f(x)\sin nx\,dx\right]$$

[using formula

$$\int_0^{2\pi}\{f(x)\}^2 dx = a_0\{2\pi a_0\} + \sum_{n=1}^{\infty} a_n^2\pi + \sum_{n=1}^{\infty} b_n^2\pi$$

$$\boxed{\int_0^{2\pi} f(x)^2 dx = 2\pi a_0^2 + \sum_{n=1}^{\infty} a_n^2\pi + \sum_{n=1}^{\infty} b_n^2\pi}$$

$$\left[\frac{1}{2\pi}\int \{f(x)\}^2 dx = a_0^2 + \sum_{n=1}^{\infty}\left[a_n^2 + b_n^2\right.\right.$$

Fourier Integral Theoram:→

If a function $f(x)$ satisfies the following conditions:-
(i) $f(x)$ is defined in interval $-L < x < L$
(ii) $f(x)$ and $f'(x)$ are continuous in interval $(-L, L)$.
(iii) $f(x)$ is periodic with period $2L$
(iv) $f(x)$ is absolutely integrable in interval $-\infty < x < \infty$.
Then Fourier integral thm states that,

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(v)\left[\int_{-\infty}^{+\infty} f \cos\{\omega(x-v)\}d\omega\right]dv$$

Proof:→ Let function $f(x)$ is expressed in form of Fourier Series as,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \quad \text{——— (I)}$$

where,

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx \quad \text{——— (II)}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{\pi n x}{L}dx \quad \text{——— (III)}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{\pi n x}{L}dx \quad \text{——— (IV)}$$

If in eqn (II), (III) and (IV), x is changed to variable 't', it does not changes RHS because a_0, a_n, b_n are independent of x.

$\therefore$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(t)dt \quad \text{——— (V)}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{\pi n t}{L}\right)dt \quad \text{——— (VI)}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{\pi n t}{L}\right)dt \quad \text{——— (VII)}$$

Using these values of a_0, a_n and b_n in eqn (I), we get,

$$f(x) = \frac{1}{2L}\int_{-L}^{L} f(t)dt + \sum_{n=1}^{\infty}\left[\frac{1}{L}\int_{-L}^{L}\left[f(t)\cos\left(\frac{\pi n t}{L}\right)dt\right]\cos\frac{\pi n x}{L} + \sum_{n=1}^{\infty}\frac{1}{L}\int_{-L}^{L}\left[f(t)\sin\frac{\pi n t}{L}dt\right]\sin\frac{\pi n x}{L}\right]$$

$$f(x)=\frac{1}{2L}\int_{-L}^{L} f(t)\,dt+\sum_{n=1}^{\infty}\frac{1}{L}\int_{-L}^{L} f(t)\left[\cos\frac{\pi n x}{L}\cos\frac{\pi n t}{L}+\sin\frac{\pi n x}{L}\sin\frac{\pi n t}{L}\right]dt$$

$$f(x)=\frac{1}{2L}\int_{-L}^{L} f(t)\,dt+\sum_{n=1}^{\infty}\frac{1}{L}\int_{-L}^{L} f(t)\cos\left[\frac{\pi n}{L}(x-t)\right]dt$$

$$f(x)=\frac{1}{2\pi}\left[\frac{\pi}{L}\int_{-L}^{L} f(t)\,dt+\sum_{n=1}^{\infty}\frac{2\pi}{L}\int_{-L}^{L} f(t)\cos\left[\frac{n\pi}{L}(x-t)\right]dt\right]$$

$$f(x)=\frac{1}{2\pi}\int_{-L}^{L} f(t)\left[\frac{\pi}{L}+\sum_{n=1}^{\infty}\frac{2\pi}{L}\cos\left[\frac{n\pi}{L}(x-t)\right]\right]dt$$

$$f(x)=\frac{1}{2\pi}\int_{-L}^{L} f(t)\left[\frac{\pi}{L}\cos\left[0\cdot\frac{\pi}{L}(x-t)\right]+\sum_{n=1}^{\infty}\frac{\pi}{L}\cos\left[\frac{n\pi}{L}(x-t)\right]+\sum_{n=1}^{\infty}\frac{\pi}{L}\cos\frac{n\pi(x-t)}{L}\right]dt$$

$[\because 2\cos\theta=\cos\theta+\cos\theta]$

$$f(x)=\frac{1}{2\pi}\int_{-L}^{L} f(t)\left[\frac{\pi}{L}\cos\left(0\cdot\frac{\pi}{L}(x-t)\right)+\sum_{n=1}^{\infty}\frac{\pi}{L}\cos\left[\frac{n\pi}{L}(x-t)\right]+\sum_{n=1}^{\infty}\frac{\pi}{L}\cos\left(\frac{-n\pi(x-t)}{L}\right)\right]dt$$

$[\because \cos\theta=\cos(-\theta)]$

$$f(x)=\frac{1}{2\pi}\int_{-L}^{L} f(t)\left[\sum_{n=0}^{\infty}\frac{\pi}{L}\cos\left(\frac{n\pi(x-t)}{L}\right)+\sum_{n=-1}^{-\infty}\frac{\pi}{L}\cos\left[\frac{n\pi(x-t)}{L}\right]\right]dt$$

$$f(x)=\frac{1}{2\pi}\int_{-L}^{L} f(t)\left[\lim_{n\to\infty}\sum_{p=-n}^{n}\frac{\pi}{L}\cos\left(\frac{\pi p(x-t)}{L}\right)\right]dt$$

When $L\to\infty$, $\frac{L}{\pi}\to\infty$, Then $\Delta\omega=\frac{\pi}{L}\to 0$

$$\lim_{L\to\infty}\sum_{p=-\infty}^{\infty}\frac{\pi}{L}\cos\left[\frac{\pi p(x-t)}{L}\right]=\lim_{\Delta\omega\to 0}\sum_{p=-\infty}^{+\infty}\Delta\omega\cos[(p(x-t)\Delta\omega]$$

$$\Rightarrow \lim_{L\to\infty}\sum_{p=-\infty}^{\infty}\frac{\pi}{L}\cos\left(\frac{\pi p(x-t)}{L}\right)=\int_{-\infty}^{+\infty}\cos[\omega(x-t)]\,d\omega \quad \{\text{By def}\}$$

Proof: $\frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x)\left[\int_{-\infty}^{+\infty}\cos(\omega(x-v))\,d\omega\right]d\omega$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(v)\left[\int_{-\infty}^{\infty} \cos(\omega(x-v)) + i\sin(\omega(x-v))\right] d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i\omega(x-v)} \, d\omega \, dv$$