Arbitrary Series

Abel's Lemma (or Abel's Inequality)

Statement : if the sequence $\langle S_n \rangle$ of partial sums of the series $\sum_{n=1}^{\infty} a_n$ satisfies $m \le S_n \le M$, $n \in N$ and $\langle b_n \rangle$ be a sequence of non increasing, non-negative real numbers, then

$$m b_1 \le \sum_{k=1}^{n} a_k b_k \le M b_1$$

Proof : Let $\langle S_n \rangle$ denote the sequence of partial sum of the series $\sum_{n=1}^{\infty} a_n$

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n$$

So that $S_1 = a_1$, $S_2 = a_1 + a_2$, $S_3 = a_1 + a_2 + a_3$, ...

$a_1 = S_1$ $\quad$ $a_2 = S_2 - a_1 = S_2 - S_1$ $\quad$ $a_3 = S_3 - (a_1 + a_2)$, $a_3 = S_3 - S_2$

Also $\quad m \le S_n \le M$, $n \in N$ —— (1) (given)

Now,

$$\sum_{k=1}^{n} a_k b_k = a_1 b_1 + a_2 b_2 + \cdots + a_{n-1} b_{n-1} + a_n b_n$$

$$= S_1 b_1 + (S_2 - S_1) b_2 + (S_3 - S_2) b_3 + \cdots + (S_{n-1} - S_{n-2}) b_{n-1} + (S_n - S_{n-1}) b_n$$

$$= S_1(b_1 - b_2) + S_2(b_2 - b_3) + \cdots + S_{n-1}(b_{n-1} - b_n) + S_n b_n \quad \text{—— (2)}$$

$$\sum_{k=1}^{n} a_k b_k \le M(b_1 - b_2) + M(b_2 - b_3) + \cdots + M(b_{n-1} - b_n) + M b_n$$ (From (1))

$$= M(b_1 - b_2 + b_2 - b_3 + \cdots + b_{n-1} - b_n + b_n)$$

$$= M b_1$$

$[S_1 \le M, S_2 \le M, \ldots, S_n \le M]$

$$\Rightarrow \sum_{k=1}^{n} a_k b_k \le M b_1 \quad \text{—— (3)}$$

Also,

$$\sum_{k=1}^{n} a_k b_k = S_1(b_1 - b_2) + S_2(b_2 - b_3) + \cdots + S_{n-1}(b_{n-1} - b_n) + S_n b_n$$

$$\ge m(b_1 - b_2) + m(b_2 - b_3) + \cdots + m(b_{n-1} - b_n) + m b_n$$

[From $m \le S_1$, $m \le S_2$ ....., $m \le S_n$]

$$= m(b_1 - b_2 + b_2 - b_3 + \cdots + b_{n-1} - b_n + b_n)$$

$$= m b_1$$

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$\Rightarrow \quad \sum_{k=1}^{n} a_k b_k \geq m b_1 \quad \text{--- (4)}$

From (3) & (4)

$$m b_1 \leq \sum_{k=1}^{n} a_k b_k \leq M b_1$$

Abel's Test

St: if $\sum_{n=1}^{\infty} a_n$ is convergent and the sequence $\langle b_n \rangle$ is monotonic and bounded, then $\sum_{n=1}^{\infty} a_n b_n$ is convergent

Proof: As the sequence $\langle b_n \rangle$ is monotonic and bounded

$\therefore$ Sequence $\langle b_n \rangle$ is convergent to b

$$b = \begin{cases} \text{glb of } \langle b_n \rangle & \text{if } \langle b_n \rangle \text{ is decreasing} \\ \text{lub of } \langle b_n \rangle & \text{if } \langle b_n \rangle \text{ is increasing} \end{cases}$$

[$\because$ a monotonically increasing sequence $\langle b_n \rangle$ which is bounded above converges to its least upper bound]

[a monotonically decreasing sequence $\langle b_n \rangle$ which is bounded below converges to its greatest lower bound]

Let us define $\quad u_n = \begin{cases} b - b_n & \text{if } \langle b_n \rangle \text{ is increasing} \\ b_n - b & \text{if } \langle b_n \rangle \text{ is decreasing} \end{cases}$

we discuss the two cases:

Case (i) Sequence $\langle b_n \rangle$ is increasing

When $\langle b_n \rangle$ is increasing, b is the lub of the sequence so that

$b_n \leq b$ for all $n \quad \Rightarrow \quad b - b_n \geq 0$

$u_n \geq 0$ for all $n \in N$ i.e u_n is non-negative (by U

and $\quad u_n - u_{n+1} = b - b_n - (b - b_{n+1}) = b_{n+1} - b_n \geq 0$

$\Rightarrow \quad u_n \geq u_{n+1} \quad \forall\, n \in N \quad$ i.e $\langle u_n \rangle$ is non increasing

Case (ii) Sequence $\langle b_n \rangle$ is decreasing:

When $\langle b_n \rangle$ is decreasing, b is the glb of the sequence so tha

$b_n \geq b \;\; \forall\, n \quad \Rightarrow \quad b_n - b \geq 0$

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So

and $u_n \geq 0 \ \forall\ n \in N$ i.e u_n is non negative (by (1))

$u_n - u_{n+1} = (b_n - b) - (b_{n+1} - b) = b_n - b_{n+1} \geq 0$

$\Rightarrow \quad u_n \geq u_{n+1} \ \forall\ n \in N$

i.e $\langle u_n \rangle$ is non increasing

$\Rightarrow$ $\langle u_n \rangle$ is non increasing sequence of non-negative number

Thus, in both cases $u_n \geq u_{n+1} \geq 0 \ \forall\ n \in N$

Now, from (1)

$$b_n = \begin{cases} b - u_n & \text{if } \langle b_n \rangle \text{ is increasing} \\ b + u_n & \text{if } \langle b_n \rangle \text{ is decreasing} \end{cases}$$

$\Rightarrow$

$$a_n b_n = \begin{cases} ba_n - a_n u_n & \text{if } \langle b_n \rangle \text{ is increasing} \\ ba_n + a_n u_n & \text{if } \langle b_n \rangle \text{ is decreasing} \end{cases} \quad -(2)$$

Since $\sum_{n=1}^{\infty} a_n$ is convergent, therefore $\sum_{n=1}^{\infty}$ ~~bn~~ ba_n is convergent —(3)

Now, we shall prove that for $\sum_{n=1}^{\infty} a_n b_n$ to be convergent, $\sum_{n=1}^{\infty} ba_n$ and $\sum_{n=1}^{\infty} a_n u_n$ is also convergent

by above eq (3) $\sum_{n=1}^{\infty} ba_n$ is convergent

Now, we shall prove that $\sum_{n=1}^{\infty} a_n u_n$ is convergent

Now since $\sum_{n=1}^{\infty} a_n$ is convergent so by Cauchy general princip of convergence, given $\varepsilon > 0$ there exist a +ve integer m such t

$$|a_{m+1} + a_{m+2} + \dots + a_n| < \varepsilon \quad \text{whenever } n \geq m$$

Using above Lemma (Abel's Lemma), we have

$$|a_{m+1} u_{m+1} + a_{m+2} u_{m+2} + \dots + a_n u_n| \leq \varepsilon u_{m+1} \leq \varepsilon u_1$$

[$\because$ $\langle u_n \rangle$ is non increasing sequence of non negative terms]

Thus, by Cauchy's general principle of convergence $\sum_{n=1}^{\infty} a_n u_n$ is con —(4)

Using (2), (3), & (4) we have $\sum_{n=1}^{\infty} a_n b_n$ is convergent.

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Cauchy's General Principle of Convergence

A necessary and sufficient condition for a series $\sum_{n=1}^{\infty} a_n$ to be convergent is that to each $\epsilon > 0$, there exists a positive integer m such that $|a_{m+1} + a_{m+2} + \cdots + a_n| < \epsilon$ whenever $n > m$.

Example 1 Show that $\sum_{n=1}^{\infty} \frac{1}{n^p} \cdot a_n$ $(p \geq 0)$ is convergent if $\sum_{n=1}^{\infty} a_n$ is convergent.

Sol Here series involve two term $\frac{1}{n^p}$ and a_n

where $\sum_{n=1}^{\infty} a_n$ is given to be convergent.

To prove $\sum_{n=1}^{\infty} \frac{1}{n^p} \cdot a_n$ to be convergent

we apply Abel's Test.

[if $\sum_{n=1}^{\infty} a_n$ is convergent and the sequence $\langle b_n \rangle$ is monotonic and bounded then $\sum_{n=1}^{\infty} a_n b_n$ is convergent]

Here $\sum_{n=1}^{\infty} a_n$ is given to be convergent

we need to prove $\sum_{n=1}^{\infty} \frac{1}{n^p}$ to be monotonic and bounded

Let $b_n = \frac{1}{n^p}$, $p \geq 0$

$$b_{n+1} = \frac{1}{(n+1)^p}$$

we have
$$n < n+1 \quad \forall n$$
$$n^p < (n+1)^p \quad \forall n$$
$$\frac{1}{n^p} > \frac{1}{(n+1)^p} \quad \forall n$$
$$b_n > b_{n+1} \quad \forall n$$
This proves b_n to be monoton

Now $\lim_{n \to \infty} b_n = \lim_{n \to \infty} \frac{1}{n^p} = 0$

Also $b_1 = 1$, $b_2 = \frac{1}{2^p}$, $b_3 = \frac{1}{3^p}$ ....

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$\therefore 0 \le b_n \le 1 \quad \forall n$

Thus, the sequence $\langle b_n \rangle$ is monotonic and bounded

$\therefore$ By Abel's Test, $\sum_{n=1}^{\infty} a_n b_n$ is convergent

Example 2 Show that the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left(1+\frac{1}{n}\right)^{-n}$ is convergent

Solution As we see that the two terms involve in the series are $\frac{(-1)^{n-1}}{n}$ and $\left(1+\frac{1}{n}\right)^{-n}$

First term $a_n = \frac{(-1)^{n-1}}{n}$ is convergent ~~cgt~~ by Leibnitz Test as we discussed in chapter 5 and we prove $b_n = \left(1+\frac{1}{n}\right)^{-n}$ is bounded and monotonic in chapter 2 (sequence)

To show that $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$ is cgt. Leibnitz's Test States that

[The alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} a_n = a_1 - a_2 + a_3 + \dots (a_n > 0) \ \forall n$ is convergent if i) $a_{n+1} \le a_n \ \forall n$ (ii) $\lim_{n\to\infty} a_n = 0$

Consider $t_n = \frac{1}{n}$ so that $\lim_{n\to\infty} t_n = 0$.

$$t_{n+1} = \frac{1}{n+1}$$

$n < n+1$ for all n

$\frac{1}{n} > \frac{1}{n+1}$ for all n

$t_n > t_{n+1}$ for all n

By Leibnitz Test, $\sum_{n=1}^{\infty} (-1)^{n-1} t_n$ is convergent

Thus $\sum_{n=1}^{\infty} a_n$ is convergent ——— (1)

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Now, $\left(1+\frac{1}{n}\right)^n = 1 + n\cdot\frac{1}{n} + \frac{n(n-1)}{2!}\cdot\frac{1}{n^2} + \cdots + \frac{1}{n^n}$

$$= 1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\cdots+\frac{1}{n^n} \quad \text{—(2)}$$

$$\left(1+\frac{1}{n+1}\right)^{n+1} = 1+1+\frac{1}{2!}\left(1-\frac{1}{n+1}\right)+\cdots+\frac{1}{(n+1)^{n+1}} \quad \text{—(3)}$$

~~From~~ As we see $n < n+1$

$$\frac{1}{n} > \frac{1}{n+1}.$$

$$\left(1-\frac{1}{n}\right) < \left(1-\frac{1}{n+1}\right) \quad \forall n.$$

From (2) & (3) we have

$$\left(1+\frac{1}{n}\right)^n < \left(1+\frac{1}{n+1}\right)^{n+1} \quad \forall n$$

$$\left(1+\frac{1}{n}\right)^{-n} > \left(1+\frac{1}{n+1}\right)^{-(n+1)} \quad \forall n$$

$$b_n > b_{n+1} \quad \forall n$$

sequence $\langle b_n \rangle$ is decreasing

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{-n} = \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e}$$

[$\because \left(1+\frac{1}{n}\right)^n = e$ This can be proved by L' Hospital rule)]

As $b_1 = \frac{1}{2}$ $\quad \frac{1}{e} \le b_n \le \frac{1}{2} \quad \forall n$

sequence $\langle b_n \rangle$ is monotonic & bounded —(4)

Using (1) & (4) and Applying Abel's Test $\sum_{n=1}^{\infty} a_n b_n$ is converg

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