Imp To Prove that $J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - x\sin\theta)\, d\theta$

for all Integral value of n.

Pf :- we know that

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = J_0(x) + J_1(x)t + J_2(x)t^2 + \cdots$$

$$\cdots + J_{-1}(x)t^{-1} + J_{-2}(x)t^{-2} + \cdots \quad (1)$$

Now $J_n(x) = (-1)^n J_{-n}(x)$

$\therefore J_{-n}(x) = (-1)^n J_n(x)$

Using it in eq$^n$ (1), we have

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = J_0(x) + J_1(x)t + J_2(x)t^2 + \cdots$$

$$\cdots - J_1(x)\frac{1}{t} + J_2(x)\frac{1}{t^2} - \cdots$$

$$= J_0(x) + \left(t - \frac{1}{t}\right)J_1(x) + \left(t^2 + \frac{1}{t^2}\right)J_2(x) + \cdots \quad (2)$$

Put $t = \cos\theta + i\sin\theta$, $\frac{1}{t} = \cos\theta - i\sin\theta$

$\therefore t - \frac{1}{t} = 2i\sin\theta$

also $t^n = \cos n\theta + i\sin n\theta$, $\frac{1}{t^n} = \cos n\theta - i\sin n\theta$

$t^n + \frac{1}{t^n} = 2\cos n\theta$, $t^n - \frac{1}{t^n} = 2i\sin n\theta$

Using all these result in eqⁿ (2), we have.

$$e^{\frac{x}{2}(2i\sin\theta)} = J_0(x) + 2i\sin\theta\, J_1(x) + 2\cos 2\theta\, J_2(x) + 2i\sin 3\theta\, J_3(x) + - - -$$

Or

$$\cos(x\sin\theta) + i\sin(x\sin\theta) = J_0(x) + 2\left[J_2(x)\cos 2\theta + J_4(x)\cos 4\theta + ---\right] + 2i\left[J_1(x)\sin\theta + ---\right]$$

$$\left[\because e^{i\theta} = \cos\theta + i\sin\theta\right.$$

Equating real and imaginary part

$$\cos(x\sin\theta) = J_0(x) + 2\left[J_2(x)\cos 2\theta + - - -\right] \quad (3)$$

$$\sin(x\sin\theta) = 2\left[J_1(x)\sin\theta + - - -\right] \quad (4)$$

Equation (3) and (4) are known as Jacobi series.



Multiplying both sides of (3) by $\cos n\theta$ and Integrating within the limit 0 to $\pi$, we have.

$$\int_0^{\pi} \cos(x\sin\theta)\cos n\theta\, d\theta = \int_0^{\pi} \left(J_0(x)\cos n\theta + 2J_2(x)\cos n\theta\cos 2\theta + ---\right) d\theta$$

$$\int_0^{\pi} \cos(x\sin\theta)\cos n\theta \, d\theta = \begin{cases} \pi J_n(x) & \text{when } n \text{ is even} \\ 0 & \text{when } n \text{ is odd.} \end{cases} \quad \text{(A)}$$

Similarly multiply eq$^n$ (4) by $\sin n\theta$ and Integrating within limit $0$ to $\pi$ we have

$$\int_0^{\pi} \sin(x\sin\theta)\sin n\theta \, d\theta = \begin{cases} 0 & \text{when } n \text{ is even} \\ \pi J_n(x) & \text{when } n \text{ is odd} \end{cases} \quad \text{(B)}$$

Adding eq$^n$ (A) and (B) we get

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \left[\cos(x\sin\theta)\cos n\theta + \sin(x\sin\theta)\sin n\theta\right] d\theta$$

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos\left(n\theta - x\sin\theta\right] d\theta .$$

This is the required result

H.P