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Normed Linear Space:- Let $V(F)$ be a V.S.
Defined norm of $u = \|u\| = \sqrt{\langle u, u\rangle}$ for all $u \in V$.

Then the vector space V with the above defined norm is called normed Linear Space iff it satisfied following cond:

(i) $\|u\| \geq 0$ and $\|u\| = 0$ iff $u = 0$

(ii) $\|\alpha u\| = |\alpha| \cdot \|u\|$, where $\alpha \in F$

(iii) $\|u + v\| \leq \|u\| + \|v\|$, where $u, v \in V$.

Any Norm satisfies above three cond. on V then it is called Normed linear Space.

EX12) Prove that every I.P.S. is a normed linear Space but converse is NOT true.

Solution:- Let $V(F)$ be an Inner Product Space.
Let $u, v \in V$ and $\alpha \in F$

Define norm $u = \|u\| = \sqrt{\langle u, u\rangle}$

Since $\langle u, u\rangle \to$ inner Product of u with u defined on V.

$\Rightarrow \|u\|$ is defined on V.

Now; to Prove V is Normed linear Space with $\|u\|$. We have to Prove above three conditions:

(i). Since $\langle u, u\rangle \geq 0$ and $\langle u, u\rangle = 0$ iff $u = 0$

By 2nd cond. of I.P.S. $\langle u, u\rangle \geq 0$ and $\langle u, u\rangle = 0$ iff $u = 0$

$\therefore \|u\| \geq 0$ and $\|u\| = 0$ iff $u = 0$.

(ii) $\|\alpha u\| = |\alpha| \cdot \|u\|$

L.H.S. $\|\alpha u\| = \sqrt{\langle \alpha u, \alpha u\rangle}$

$\Rightarrow \|\alpha u\|^2 = \langle \alpha u, \alpha u\rangle$

$= \alpha \langle u, \alpha u\rangle$

$= \alpha \bar{\alpha} \langle u, u\rangle$

$\because \langle \alpha u, v\rangle = \alpha \langle u, v\rangle$

$\langle u, \alpha v\rangle = \bar{\alpha} \langle u, v\rangle$

$$\|\alpha u\|^2 = \alpha\bar{\alpha}\langle u,u\rangle = |\alpha|^2\|u\|^2$$

E:

$$\Rightarrow \|\alpha u\| = |\alpha|\cdot\|u\|$$

$\because z\bar{z} = |z|^2$

So (III) $\|u+v\| \le \|u\| + \|v\|$

L.H.S $\Rightarrow$ $\|u+v\| = \sqrt{\langle u+v, u+v\rangle}$

$$\Rightarrow \|u+v\|^2 = \langle u+v, u+v\rangle$$
$$= \langle u, u+v\rangle + \langle v, u+v\rangle$$
$$= \langle u,u\rangle + \langle u,v\rangle + \langle v,u\rangle + \langle v,v\rangle$$
$$= \|u\|^2 + \langle u,v\rangle + \overline{\langle u,v\rangle} + \|v\|^2$$
$$= \|u\|^2 + \|v\|^2 + 2\,\mathrm{Re}\langle u,v\rangle$$
$$\le \|u\|^2 + \|v\|^2 + 2|\langle u,v\rangle|$$
$$\le \|u\|^2 + \|v\|^2 + 2\|u\|\|v\|$$

{By cauchy schwarz inequality}

{By Ist cond of I.P.S. $\langle u,v\rangle = \overline{\langle v,u\rangle}$ OR $\langle v,u\rangle = \overline{\langle u,v\rangle}$

$z+\bar{z} = 2\,\mathrm{Re}\,z$. Here $z = \langle u,v\rangle$

$z = x+iy$

$\mathrm{Re}\,z \le |z|$

$\Rightarrow x \le \sqrt{x^2+y^2}$

$\Rightarrow x^2 \le x^2+y^2$

$$\|u+v\|^2 \le (\|u\|+\|v\|)^2$$

$$\Rightarrow \|u+v\| \le \|u\|+\|v\|.$$

$\therefore$ V Satisfies the all the three cond. of Normed linear space. So it is N.L.S. with $\|\cdot\|$ (Norm).

Converse of above is NOT TRUE.

ie N.L.S $\not\Rightarrow$ I.P.S.

For a normed vector space $R^2(R)$ with a norm defined by $\|u\| = \max(|u_1|, |u_2|)$, Where $u = (u_1, u_2)$

Here; it is NOT possible to defined an Inner Product $\langle\,,\rangle$ on $R^2(R)$ such that $\langle u,u\rangle = \|u\|^2$

Hence; every normed vector space is NOT an I.P.S.

orthogonal vectors and orthogonal Complement :-

1) orthogonal vectors :- Let V be an I.P.S. A vector $u \in V$ is s.t.b. orthogonal to vector $v \in V$ (written as $u \perp v$) iff $\langle u, v \rangle = 0$

i.e iff $\langle u, v \rangle = 0 \Rightarrow u \perp v$.

2) orthogonal complement :- Let W be a subspace of an I.P.S. V. Then the set $W^{\perp} = \{ v \in V \mid \langle v, w \rangle = 0 \text{ for all } w \in W \}$

(isko W perp bolte h).

is called orthogonal complement of W.

Note :- Let W be a non-empty subset of an Inner Product $V(F)$. Then $W^{\perp}$ is a subspace of $V(F)$ and

$$(W^{\perp})^{\perp} = \{ x \in V \mid \langle x, y \rangle = 0 \text{ for all } y \in W^{\perp} \}$$

$(W^{\perp})^{\perp}$ is written as $W^{\perp\perp}$ and is known as orthogonal complement of orthogonal complement $W^{\perp}$.

IMP

Thm Let W be a subspace of an Inner Product Space $V(F)$. If $\{w_1, w_2, \text{---}, w_n\}$ is a basis of W, then $w \in W^{\perp}$ iff $\langle w, w_i \rangle = 0$; $1 \le i \le n$

Proof :-

Let $S = \{w_1, w_2, \text{---}, w_n\}$ be a basis of W. and W is a subspace of I.P.S. $V(F)$.

Let $w \in W^{\perp}$

T.P $\langle w, w_i \rangle = 0 \quad \forall \ 1 \le i \le n.$

Since S is a basis of W.

$\Rightarrow \quad w_i \in W \quad \forall \quad 1 \le i \le n$

Now, Since $w \in W^{\perp}$

By def. of $W^{\perp} \Rightarrow \langle w, w' \rangle = 0$

$\forall\ w' \in W$. — (1)

Since $w_i \in W \quad \forall \quad 1 \le i \le n$.

$\therefore$ From (1) $\Rightarrow \langle w, w_i \rangle = 0 \quad \forall \quad 1 \le i \le n$.

which is the Required Result.

Conversely; given; $\langle w, w_i \rangle = 0 \quad \forall \quad 1 \le i \le n$

T.P $\quad w \in W^{\perp}$

Since S is a basis of W; so any $w' \in W$ can be written as L.C. of elts of S.

$\therefore \quad w' = \alpha_1 w_1 + \alpha_2 w_2 + \text{———} + \alpha_n w_n \ ; \ \alpha_i \in F$

$$w' = \sum_{i=1}^{n} \alpha_i w_i$$

$$\therefore \langle w, w' \rangle = \langle w, \sum_{i=1}^{n} \alpha_i w_i \rangle$$

$$= \langle w, \alpha_1 w_1 + \alpha_2 w_2 + \text{———} + \alpha_n w_n \rangle$$

$$= \bar{\alpha}_1 \langle w, w_1 \rangle + \bar{\alpha}_2 \langle w, w_2 \rangle + \text{———} + \bar{\alpha}_n \langle w, w_n \rangle$$

$$\boxed{\because \langle w, \alpha w' \rangle = \bar{\alpha} \langle w, w' \rangle}$$

$$= \sum_{i=1}^{n} \bar{\alpha}_i \langle w, w_i \rangle$$

$$= \sum_{i=1}^{n} \bar{\alpha}_i \cdot 0 = 0$$

$\Big\{ \because \langle w, w_i \rangle = 0$ (given)

$\Rightarrow \langle w, w' \rangle = 0 \quad \forall\ w' \in W$.

$\Rightarrow \quad w \in W^{\perp}$ [By def. of $W^{\perp}$]

Example-15 Let M and N be subspaces of a finite dimensional inner product space V. Then show that $(M+N)^{\perp} = M^{\perp} \cap N^{\perp}$

Solution:- T.P $(M+N)^{\perp} = M^{\perp} \cap N^{\perp}$

R.T.P (i) $(M+N)^{\perp} \subset M^{\perp} \cap N^{\perp}$

(ii) $M^{\perp} \cap N^{\perp} \subset (M+N)^{\perp}$.

(i) Let $z \in (M+N)^{\perp}$ — (1)

$\Rightarrow \langle z, u \rangle = 0$ for all $u \in (M+N)$ [By def.]

$\Rightarrow \langle z, x+y \rangle = 0$ for all $u = x+y \in (M+N)$ s.t. $x \in M, y \in N$.

$\Rightarrow \langle z, x \rangle + \langle z, y \rangle = 0$ where $x \in M, y \in N$.

$\Rightarrow \langle z, x \rangle = 0$ for $x \in M$

and $\langle z, y \rangle = 0$ for $y \in N$.

$\therefore \langle z, x \rangle = 0$ for all $x \in M$.

$\Rightarrow z \in M^{\perp}$

Also $\langle z, y \rangle = 0$ for all $y \in N$

$\Rightarrow z \in N^{\perp}$

$\therefore z \in M^{\perp}$ and $z \in N^{\perp} \Rightarrow z \in M^{\perp} \cap N^{\perp}$

$\therefore z \in (M+N)^{\perp} \Rightarrow z \in M^{\perp} \cap N^{\perp}$ — (2)

$\Rightarrow (M+N)^{\perp} \subset M^{\perp} \cap N^{\perp}$ — (3)

(ii) Let $z \in M^{\perp} \cap N^{\perp}$ — (4)

$\Rightarrow z \in M^{\perp}$ and $z \in N^{\perp}$

If $z \in M^{\perp} \Rightarrow \langle z, x \rangle = 0$ for all $x \in M$. [By def. of $M^{\perp}$]

and if $z \in N^{\perp}$

$\Rightarrow \langle z, y \rangle = 0$ for all $y \in N$

Here $\langle z, x \rangle = 0$, $\langle z, y \rangle = 0$ for all $x \in M, y \in N$

$\therefore \langle z, x \rangle + \langle z, y \rangle = 0$ for all $x \in M$, $y \in N$

$\Rightarrow \langle z, x+y \rangle = 0$ for all $x+y \in M+N$

$\Rightarrow z \in (M+N)^{\perp}$ [By def of $(M+N)^{\perp}$] (5)

From (4) & (5) $\Rightarrow M^{\perp} \cap N^{\perp} \subset (M+N)^{\perp}$ — (6)

From (3) & (6) $(M+N)^{\perp} = M^{\perp} \cap N^{\perp}$

Example-16) Let S be a subset of I.P.S. V.
Then show that $S^{\perp} = S^{\perp\perp\perp}$

Solution:- We know that $S^{\perp} = \{x \in V \mid \langle x, y \rangle = 0 \text{ for all } y \in S\}$ — (1)

$S^{\perp\perp} = (S^{\perp})^{\perp} = \{x \in V \mid \langle x, y \rangle = 0 \text{ for all } y \in S^{\perp}\}$ (2)

Let $x \in S$ be any arb. element.

$\Rightarrow \langle x, y \rangle = 0 \quad \forall\, y \in S^{\perp}$ [$\because$ S and $S^{\perp}$ are orthogonal subspaces]

$\Rightarrow x \in S^{\perp\perp}$ [By (2)]

$\therefore x \in S \Rightarrow x \in S^{\perp\perp}$ $\Rightarrow S \subseteq S^{\perp\perp}$ — (3)

Also; By property of orthogonal complement

If $W_1 \subset W_2 \Rightarrow W_2^{\perp} \subset W_1^{\perp}$ where $W_1, W_2 \subset V$.

$\therefore (3) \Rightarrow S^{\perp\perp\perp} \subseteq S^{\perp}$ — (4)

Also; (3) is true for any subset S of V.

Now, $S^{\perp} \subset V$. So it is true for $S^{\perp}$ also

$\therefore$ (3) $\Rightarrow S^{\perp} \subseteq S^{\perp\perp\perp}$ (5) [Replace S to $S^{\perp}$]

From (4) & (5) $S^{\perp} = S^{\perp\perp\perp}$