Subject :→ Vector Calculus

Class - B.Sc I

Topic :→ Vector Integration

Unit - 4

Indefinite Integral :→ Let $\vec{f}(t)$ and $\vec{F}(t)$ be the two vector functions of a scalar variable t such that

$$\frac{d}{dt}(\vec{F}(t)) = \vec{f}(t), \text{ then}$$

$\vec{F}(t)$ is called an indefinite integral of $\vec{f}(t)$ w.r.t t and is written as,

$$\int \vec{f}(t)\, dt = \vec{F}(t)$$

Here, the function $\vec{f}$, which is to be integrated is called the integrand.

Definite Integral :→

If $\frac{d\vec{F}(t)}{dt} = \vec{f}(t)$ $\forall$ values of t in $[a, b]$, then the definite integral between the limits $t = a$ and $t = b$ is defined as,

$\int_a^b \vec{f}(t)\, dt$ and is defined as,

$$\int^b \vec{f}(t)\, dt = \vec{F}(b) - \vec{F}(a)$$

Some Examples Based on Integration :→

Exp ① ⇒ The acceleration of a particle at any time $t \geq 0$ is given by,

$$\vec{a} = 18\cos 3t\,\hat{i} - 8\sin 2t\,\hat{j} + 6t\,\hat{k}$$

If the velocity $\vec{v}$ and displacement $\vec{r}$ be zero at $t=0$, find $\vec{v}$ and $\vec{r}$ at any time.

Solⁿ ⇒ Here, acceleration is

$$\vec{a} = \frac{d\vec{v}}{dt} = 18\cos 3t\,\hat{i} - 8\sin 2t\,\hat{j} + 6t\,\hat{k} \quad \longrightarrow ①$$

We have to find velocity $\vec{v}$,
so, integrating ① w.r.t t, we get

$$\vec{v} = 6\sin 3t\,\hat{i} + 4\cos 2t\,\hat{j} + 3t^2\,\hat{k} + c \quad \longrightarrow ②$$

where, c is a constant of integration.

Now, it is given that $v=0$ at $t=0$
Using this in eqn. ②, we get

$$0 = 4\hat{j} + c$$

$$\Rightarrow \boxed{c = -4\hat{j}}$$

so, by ②,

$$\boxed{\vec{v} = 6\sin 3t\,\hat{i} + (4\cos 2t - 4)\hat{j} + 3t^2\,\hat{k}}$$

which is required velocity.

Now, $\vec{v} = \frac{d\vec{r}}{dt} = 6\sin 3t\,\hat{i} + (4\cos 2t - 4)\,\hat{j} + 3t^2\hat{k} \rightarrow (3)$

For finding $\vec{r}$, integrating eqn. (3) w.r.t t, we get

$$\vec{r} = \hat{i}\int 6\sin 3t\,dt + \hat{j}\int (4\cos 2t - 4)\,dt + \hat{k}\int 3t^2\,dt$$

$$\vec{r} = -2\cos 3t\,\hat{i} + (2\sin 2t - 4t)\,\hat{j} + t^3\hat{k} + c_1 \rightarrow (4)$$

where, c_1 is a const of integration.

Now, it is given that $\vec{r} = 0$ at $t = 0$. Using this in eqn. (4), we get

$$0 = -2\hat{i} + c_1$$

$$\Rightarrow \boxed{c_1 = 2\hat{i}}$$

Using value of c_1 in eqn. (4), we get

$$\vec{r} = -2\cos 3t\,\hat{i} + (2\sin 2t - 4t)\,\hat{j} + t^3\hat{k} + 2\hat{i}$$

$$\boxed{\vec{r} = 2(1 - \cos 3t)\,\hat{i} + (2\sin 2t - 4t)\,\hat{j} + t^3\hat{k}}$$

which is the required position vector.

Date:
Page:

Que ⇒ Given that $\vec{r} = \begin{cases} 2\hat{i} - \hat{j} + 2\hat{k}, & t=2 \\ 4\hat{i} - 2\hat{j} + 3\hat{k}, & t=3 \end{cases}$

prove that,

$$\int_2^3 \left(\vec{r} \cdot \frac{d\vec{r}}{dt}\right) dt = 10.$$

Solⁿ ⇒ As, $\frac{d}{dt}(\vec{f}^2) = 2\vec{f}\,\frac{d\vec{f}}{dt}$ $\left[\frac{d}{dt}(\vec{f}\cdot\vec{f}) = \vec{f}\cdot\frac{d\vec{f}}{dt} + \frac{d\vec{f}}{dt}\cdot\vec{f}\right]$

$$\Rightarrow \int_a^b \left(2\vec{f}\,\frac{d\vec{f}}{dt}\right) = \left[\vec{f}^2\right]_a^b$$

or,

$$\int_a^b \vec{f}\,\frac{d\vec{f}}{dt} = \frac{1}{2}\left[\vec{f}^2\right]_a^b$$

Using this result, we can write

$$\int_2^3 \left(\vec{r}\;\frac{d\vec{r}}{dt}\right)dt = \frac{1}{2}\left[\vec{r}^2\right]_2^3 \longrightarrow (1)$$

Now, for $t=2$, $\vec{r} = 2\hat{i} - \hat{j} + 2\hat{k}$
so, $(\vec{r}^2)_{t=2}$ is

$\vec{r}^2 = \vec{r}\cdot\vec{r} = 4+1+4 = 9$ at $t=2$

||ly, $(\vec{r})^2 = \vec{r}\cdot\vec{r} = (4\hat{i} - 2\hat{j} + 3\hat{k})\cdot(4\hat{i} - 2\hat{j} + 3\hat{k})$

$= 16 + 4 + 9 = 29$

at $t=3$

Using these values of $\vec{r}^2$ at $t=3$ and $t=2$ in eqn. ①, we get

$$\int_2^3 \left(\vec{r} \cdot \frac{d\vec{r}}{dt}\right) dt = \frac{1}{2}\left[29-9\right] = 10$$

Thus, $\boxed{\int_2^3 \left(\vec{r} \cdot \frac{d\vec{r}}{dt}\right) dt = 10}$

Scanned with CamScanner