B.A / B.Sc (III)

Sub :→ Real and Complex Analysis

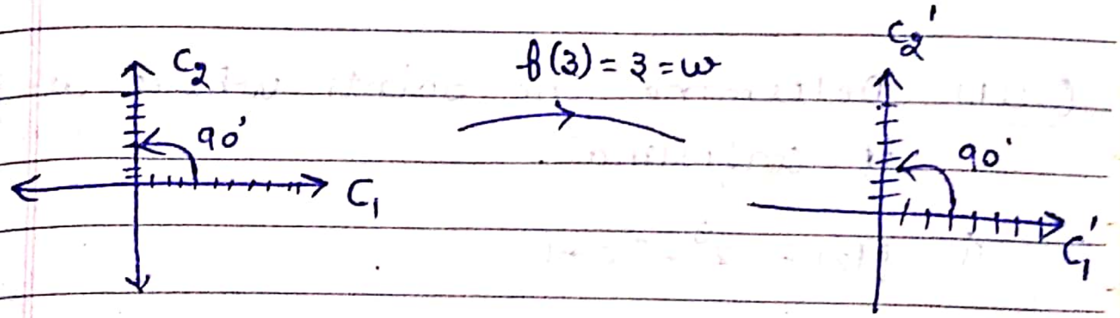
Mobius Transformation

Unit - 4

Chapter - 6

Conformal Mapping :→ A mapping $f(z)$ is said to be conformal, if it preserves the sense of rotation as well as the magnitude of the angle, i.e, after transformation from z-plane to w-plane, angle of rotation as well as well as sense of rotation remains the same.

For exp, Let $f(z) = w = z$, then



(Z Plane) (w plane)

Here, clearly after the transformation (z plane to w plane), magnitude of angle (i.e, 90°) also the sense of rotation (i.e, anticlockwise) both are preserved.

Hence, mapping is conformal.

The necessary and sufficient condⁿ for the transformation $f(z)$ to be conformal is that

(i) $f(z)$ is analytic.

(ii) $f'(z) \neq 0$.

i.e, If

(i) $f(z)$ is analytic in a domain D containing z_0

and, (ii) $f'(z_0) \neq 0$,

then, $f(z)$ is conformal mapping at z_0.

Ques1) Determine the points where $w = f(z)$ is conformal,

(i) $f(z) = z^3 - 3z + 1$

Solⁿ) Clearly, ① $f(z)$ is analytic in $\mathbb{C}$.

② Find $f'(z)$ and equate it to 0.

$f'(z) = 0$

$\Rightarrow \quad 3z^2 - 3 = 0$

$\Rightarrow \quad \boxed{z = \pm 1}$

But, for $f(z)$ to be conformal, $f'(z) \neq 0$

So, for $z = +1$ and -1,

$f'(z) = 0$

$\Rightarrow$ For these points, $f(z)$ is not conformal.

So,

$f(z)$ is conformal at $\mathbb{C} - \{-1, 1\}$.

Que2) If $u = 2x^2 + y^2$, $v = \frac{y^2}{x}$, then

Prove that the transformation $f(z) = w = u + iv$ is not conformal.

Soln) For, $f(z) = w = u + iv$ to be conformal, we should have

① $f(z)$ analytic

② $f'(z) \neq 0$

Checking ① :- Here, $u_x = 4x$; $u_y = 2y$

$v_x = -\frac{y^2}{x^2}$, $v_y = \frac{2y}{x}$

Clearly, $u_x \neq v_y$

and $u_y \neq -v_x$

$\Rightarrow$ C-R eqns. are not satisfied.

$\Rightarrow$ The transformation $f(z) = w$ is not analytic

$\Rightarrow$ Not conformal.

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Coefficient of Magnification :→

Coefficient of magnification for the conformal transformation $w = f(z)$ at $z = \alpha + i\beta$ is given by,

$$|f'(\alpha + i\beta)|$$

Angle of rotation :→

Angle of rotation for the conformal transformation, $w = f(z)$ at $z = \alpha + i\beta$ is,

$$\arg|f'(\alpha + i\beta)|$$

Que ⇒ Find the coefficient of magnification and angle of rotation at $z = 2+i$ for the transformation $w = z^2$.

Sol:→

$$w = z^2 = f(z)$$

Then, $f'(z) = 2z$

So, $|f'(z)|_{z=2+i}$ = Coeff of Magnificatn

Hence, coeff of mag. $= |f'(z)|_{z=2+i}$

$$= |2(2+i)| = |4+2i|$$

$$= \sqrt{16+4} = \sqrt{20} = \boxed{2\sqrt{5}}$$

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Angle of rotation $= \arg|f'(z)|$

$= \arg(4+2i)$

$= \tan^{-1}\frac{2}{4}$

$= \tan^{-1}\frac{1}{2}$

* **Linear Transformation :→** A transformation of the type, $w = \alpha z + \beta$ (α and β are complex constants) is said to be a linear transformation.

Clearly, this transformation is the resultant of magnification, rotation and translation.

* **Mobius transformation or Bilinear Transformation :→**

A transformation of the form,

$$w = \frac{az+b}{cz+d}$$ is called a

Bilinear transformation, where a, b, c and d are complex constants.

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$$w = \frac{az+b}{cz+d}$$

Clearly, $cwz + wd - az - b = 0$

It is linear in both w and z, so it is called bilinear.