# Inner Product Spaces

**Definition** :- Let $V$ be a V.S. over a field $F$. Let $a, b \in F$ and $u, v, w \in V$.
The V.S. 'V' is s.t.b. Inner Product Space (I.P.S.) over $F$ iff $\exists$ a function $\langle , \rangle : V \times V \to F$ satisfying the following conditions

(i) $\langle u, v \rangle = \overline{\langle v, u \rangle}$ [complex conjugate of $\langle v, u \rangle$]

(ii) $\langle u, u \rangle \geq 0$ and $\langle u, u \rangle = 0$ iff $u = 0$

(iii) $\langle au + bv, w \rangle = a\langle u, w \rangle + b\langle v, w \rangle$

If the function $\langle , \rangle$ satisfies all three cond. then it is called Inner Product on V.

**Example I** - Let $u = (a_1, a_2, \ldots, a_n)$ and $v = (b_1, b_2, \ldots, b_n)$ be arbitrary elts of $C^n$. Define $\langle u, v \rangle = u \cdot v = a_1\bar{b}_1 + a_2\bar{b}_2 + \cdots + a_n\bar{b}_n$. Show that $C^n$ is I.P.S.

**Solution**:- We will prove all the properties

(i) $\langle u, v \rangle = \overline{\langle v, u \rangle}$

R.H.S $\Rightarrow \overline{\langle v, u \rangle} = \overline{(b_1\bar{a}_1 + b_2\bar{a}_2 + \cdots + b_n\bar{a}_n)}$

$= \bar{b}_1 \bar{\bar{a}}_1 + \bar{b}_2\bar{\bar{a}}_2 + \cdots + \bar{b}_n \bar{\bar{a}}_n$

$= \bar{b}_1 a_1 + \bar{b}_2 a_2 + \cdots + \bar{b}_n a_n$

$= a_1\bar{b}_1 + a_2\bar{b}_2 + \cdots + a_n\bar{b}_n$

$= \langle u, v \rangle = L.H.S$

$\left[\begin{array}{l} a_1 = x + iy \\ \bar{a}_1 = x - iy \\ \bar{\bar{a}}_1 = x + iy \\ \quad = a_1 \end{array}\right.$

(ii) $\langle u, u \rangle = a_1\bar{a}_1 + a_2\bar{a}_2 + \cdots + a_n\bar{a}_n$

$= |a_1|^2 + |a_2|^2 + \cdots + |a_n|^2$

$\geq 0$

$\left\{\begin{array}{l} z = x + iy \\ \bar{z} = x - iy \\ z \cdot \bar{z} = x^2 + y^2 \\ \quad = |z|^2 \end{array}\right.$

Also, $\langle u, u \rangle = 0$

iff $|a_1|^2 + |a_2|^2 + \cdots + |a_n|^2 = 0$

iff $|a_1| = 0, |a_2| = 0, \text{---} |a_n| = 0$

iff $a_1 = a_2 = \text{---} = a_n = 0$

iff $u = (a_1, a_2, \text{---} a_n)$

$= (0, 0, \text{---}, 0) = 0$

(iii) $\langle au + bv, w \rangle = (aa_1 + bb_1)\overline{c_1} + (aa_2 + bb_2)\overline{c_2} + \text{---} + (aa_n + bb_n)\overline{c_n}$

Where $w = (c_1, c_2, \text{---}, c_n)$

$= a(a_1\overline{c_1} + a_2\overline{c_2} + \text{---} + a_n\overline{c_n})$
$+ b(b_1\overline{c_1} + b_2\overline{c_2} + \text{---} + b_n\overline{c_n})$

$= a\langle u, w \rangle + b\langle v, w \rangle$

$\therefore$ All properties of I.P.S. are satisfied

$\therefore$ $\mathbb{C}^n$ is an I.P.S

EX 2) Let V be the v.s. over $\mathbb{C}$ of all complex valued continuous functions on interval $[\alpha, \beta]$. Define the Inner Product on V as $\langle f(t), g(t) \rangle = \int_\alpha^\beta f(t)\overline{g(t)}\, dt$. Prove that V is I.P.S.

Solution:-

(i) $\langle f(t), g(t) \rangle = \overline{\langle g(t), f(t) \rangle}$

R.H.S $\Rightarrow$ $\overline{\langle g(t), f(t) \rangle} = \overline{\left[\int_\alpha^\beta g(t)\overline{f(t)}\, dt\right]}$

$= \int_\alpha^\beta \overline{g(t)\overline{f(t)}}\, dt$

$= \int_\alpha^\beta \overline{g(t)}\, f(t)\, dt = \int_\alpha^\beta f(t)\overline{g(t)}\, dt$

$= \langle f(t), g(t) \rangle = L.H.S.$

(ii) $\langle f(t), f(t) \rangle \geq 0$

L.H.S. $\langle f(t), f(t) \rangle = \int_{\alpha}^{\beta} f(t) \cdot \overline{f(t)} \, dt = \int_{\alpha}^{\beta} |f(t)|^2 dt \geq 0$

Also $\langle f(t), f(t) \rangle = 0 \Leftrightarrow \int_{\alpha}^{\beta} |f(t)|^2 dt = 0$

$\Leftrightarrow |f(t)|^2 = 0 \Leftrightarrow |f(t)| = 0$

$\Leftrightarrow f(t) = 0$

(iii) $\langle a f(t) + b g(t), h(t) \rangle = \int_{\alpha}^{\beta} [a f(t) + b g(t)] \overline{h(t)} \, dt$

$= a \int_{\alpha}^{\beta} f(t) \overline{h(t)} \, dt + b \int_{\alpha}^{\beta} g(t) \overline{h(t)} \, dt$

$= a \langle f(t), h(t) \rangle + b \langle g(t), h(t) \rangle$

Thus V Satisfies all Properties

$\therefore$ V is an I.P.S.

**Norm of a vector** :- Let $V$ be an I.P.S. For any $v \in V$, the norm of $v$ is defined as $\sqrt{<v,v>}$ and is denoted by $\|v\|$

i.e. $\|v\| = \sqrt{<v,v>}$

or $\|v\|^2 = <v,v> \geq 0$

NOTE:-

If Norm of any vector 'v' is one then vector 'v' is called unit vector. i.e $\|v\| = 1$.

EX- Find the norm of the vector $u=(2,-3,6)$ and normalize this vector.

Sol:- Let $u = (2,-3,6)$

Then $\|u\|^2 = <u,u>$

$= <(2,-3,6),(2,-3,6)>$

$= 2^2+(-3)^2+6^2 = 49$

$\|u\| = \sqrt{49} = 7$

The corresponding Normalized vector is $\frac{u}{\|u\|}$

$= \frac{1}{7}(2,-3,6) = \left(\frac{2}{7}, \frac{-3}{7}, \frac{6}{7}\right)$

IMP

⊛ **Cauchy Schwarz Inequality** :- Let $V$ be an I.P.S. Then

$|<u,v>| \leq \|u\|\|v\|$ for all $u,v \in V$

Proof:-

If $u=0$, Then $<u,v> = <0,v> = 0$

and $\|u\| = \sqrt{<u,u>} = \sqrt{<0,0>} = 0$

IIIly; If $v=0$ Then triangular inequality holds i.e $|<u,v>| \leq \|u\| \|v\|$

Let $u \neq 0$, then $\|u\| \neq 0$ as If $\|u\| = 0$

$\Rightarrow \sqrt{<u,u>} = 0 = \sqrt{<0,0>}$

$\Rightarrow u = 0$

Let $w = v - \frac{\langle v,u\rangle u}{\|u\|^2}$ —(1)

Then $\langle w,u\rangle = \langle (v - \frac{\langle v,u\rangle}{\|u\|^2} u, u)\rangle$

$= \langle v,u\rangle - \langle (\frac{\langle v,u\rangle u}{\|u\|^2}), u\rangle$

$= \langle v,u\rangle - \frac{\langle v,u\rangle}{\|u\|^2} \langle u,u\rangle$ $\left\{\because \langle au,u\rangle = a\langle u,u\rangle\right.$

$= \langle v,u\rangle - \frac{\langle v,u\rangle}{\|u\|^2} \cdot \|u\|^2$

$= \langle v,u\rangle - \langle v,u\rangle = 0$

$\therefore \langle w,u\rangle = 0$ —(2)

Also; $\|w\|^2 = \langle w,w\rangle = \langle (v - \frac{\langle v,u\rangle}{\|u\|^2} u, w)\rangle$

$= \langle v,w\rangle - \frac{\langle v,u\rangle}{\|u\|^2} \langle u,w\rangle$

$= \langle v,w\rangle - \frac{\langle v,u\rangle}{\|u\|^2} \overline{\langle w,u\rangle}$ $\left\{\text{By Prop of I.P.S. } \langle u,w\rangle = \overline{\langle w,u\rangle}\right.$

$= \langle v,w\rangle - \frac{\langle v,u\rangle}{\|u\|^2} \cdot \overline{0}$ $\left\{\because \langle w,u\rangle = 0 \text{ By (2)}\right.$

$= \langle v,w\rangle - 0$

$\|w\|^2 = \langle v,w\rangle$

$= \langle v, (v - \frac{\langle v,u\rangle}{\|u\|^2} u)\rangle$ [By (1)]

$= \langle v,v\rangle - \frac{\overline{\langle v,u\rangle}}{\|u\|^2} \langle v,u\rangle$ $\left\{\because \langle v,\alpha w\rangle = \overline{\alpha}\langle v,w\rangle\right.$

$\|w\|^2 = \|v\|^2 - \frac{|\langle v,u\rangle|^2}{\|u\|^2}$ —(3)

[$\because \overline{z} z = |z|^2$ where $z = \langle v,u\rangle$]

Now; Since $\|w\| \geq 0$

$\Rightarrow \|w\|^2 \geq 0$

From 3 $\Rightarrow$ $$\|v\|^2 - \frac{|\langle v,u\rangle|^2}{\|u\|^2} \geq 0$$

$$\Rightarrow \frac{\|v\|^2\|u\|^2 - |\langle v,u\rangle|^2}{\|u\|^2} \geq 0$$

$$\Rightarrow \|v\|^2\|u\|^2 - |\langle v,u\rangle|^2 \geq 0$$

$$\Rightarrow \|v\|^2\|u\|^2 \geq |\langle v,u\rangle|^2$$

$$\Rightarrow \|v\|\,\|u\| \geq |\langle v,u\rangle|$$

$$\Rightarrow |\langle v,u\rangle| \leq \|u\|\,\|v\|.$$

Which is the Required Result.

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**Remark 1)** The above inequality will become an equality iff $u, v$ are Linearly dependent.

**Proof:-** Let $|\langle u, v\rangle| = \|u\|\,\|v\|$

For $u=0$ Then $u = 0\cdot v \Rightarrow u, v$ are L.D.

If $u \neq 0$, then as by above thm.

$\langle w, w\rangle = 0 \Rightarrow w = 0$ $\left\{\because \langle w, w\rangle \geq 0 \text{ and } \langle w, w\rangle = 0 \text{ iff } w = 0\right.$

on putting value of '$w$' by above thm.

$\Rightarrow v - \dfrac{\langle v, u\rangle}{\|u\|^2} u = 0$

$\Rightarrow v = \dfrac{\langle v, u\rangle}{\|u\|^2} u$ $\Rightarrow v = \alpha u$ where $\alpha = \dfrac{\langle v, u\rangle}{\|u\|^2}$

$\Rightarrow u, v$ are L.D.

**Conversely;** Let $u, v$ are L.D.

i.e $u = \alpha v$ ; $\alpha \in F$

Now $|\langle u, v\rangle| = |\langle \alpha v, v\rangle| = |\alpha \langle v, v\rangle|$

$= |\alpha|\,\|v\|^2$ — (3) $\left[\because \langle v, v\rangle = \|v\|^2\right]$

Also

$\|u\|\|v\| = \|\alpha v\|\,\|v\| = |\alpha|\,\|v\|\,\|v\|$ $\left[\because \|\alpha v\| = |\alpha|\,\|v\|\right]$

$= |\alpha|\,\|v\|^2$ — (4)

From (3) & (4) $|\langle u, v\rangle| = \|u\|\,\|v\|$ Hence the Result.

## * TRIANGLE INEQUALITY :- Let V be an I.P.S. Then

$$\|u+v\| \le \|u\| + \|v\|$$

Proof :- $\|u+v\|^2 = \langle u+v, u+v \rangle = \langle u, u+v \rangle + \langle v, u+v \rangle$

$= \langle u,u \rangle + \langle u,v \rangle + \langle v,u \rangle + \langle v,v \rangle$

$= \|u\|^2 + \underbrace{\langle u,v \rangle + \overline{\langle u,v \rangle}} + \|v\|^2$

$= \|u\|^2 + \|v\|^2 + 2\,\mathrm{Re}\langle u,v \rangle$

$\le \|u\|^2 + \|v\|^2 + 2\,|\langle u,v \rangle|$

$\le \|u\|^2 + \|v\|^2 + 2\,\|u\|\,\|v\|$

$= (\|u\| + \|v\|)^2$

Thus $\|u+v\|^2 \le (\|u\| + \|v\|)^2$

$\Rightarrow \|u+v\| \le \|u\| + \|v\|$

which is triangular inequality.

Using Prop of I.P.S.

$\langle u,v \rangle = \overline{\langle v,u \rangle}$

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$\langle v,u \rangle = \overline{\langle u,v \rangle}$

If $z = x + iy$

$\bar{z} = x - iy$

$z + \bar{z} = 2x$

$= 2\,\mathrm{Re}\,z$

$|z| = \sqrt{x^2+y^2}$

$\mathrm{Re}\,z \le |z|$

Take

$\langle u,v \rangle = z$

I.D.P

E.D.P

$G_1 \oplus G_2 \oplus G_3 \oplus G_n$

or $G_1 \times G_2 \times$

$|z-3| <$

**Remark: 2)** In an I.P.S.; If $\|u+v\| = \|u\| + \|v\|$, then the vectors $u, v$ are Linearly dependent.

**Proof:-** given:- $\|u+v\| = \|u\| + \|v\|$.

squaring b.s. $\Rightarrow$ $\|u+v\|^2 = (\|u\| + \|v\|)^2$

$= \|u\|^2 + \|v\|^2 + 2\|u\| \cdot \|v\|$

OR

$\|u\|^2 + \|v\|^2 + 2\, Re\, \langle u, v \rangle = \|u\|^2 + \|v\|^2 + 2\|u\| \cdot \|v\|$

$\Rightarrow$ $Re \langle u, v \rangle = \|u\| \|v\|$

$\Rightarrow$ $\|u\| \|v\| = Re \langle u, v \rangle$

$\leq |\langle u, v \rangle|$ — ①

[By the above thm $\|u+v\|^2 = \|u\|^2 + \|v\|^2 + 2 Re \langle u, v \rangle$; $Re \langle u, v \rangle \leq |\langle u, v \rangle|$]

ALSO

$\|u\| \|v\| \geq |\langle u, v \rangle|$ — ②

[By Schwarz Inequality]

From ① & ②, we have

$\|u\| \|v\| = |\langle u, v \rangle|$.

$\Rightarrow$ $u$ & $v$ are Linearly dependent by Remark 1.

**NOTE:-** Converse of above Remark is NOT TRUE.

**C.E:-** Let $u = (1, 0, -1)$, $v = (-2, 0, 2) \in V_3(R)$

Here $v = -2u$ $\Rightarrow$ $u, v$ are L.D.

But $\|u+v\| \neq \|u\| + \|v\|$.

Now $\|u\| = \sqrt{\langle u, u \rangle} = \sqrt{\langle (1,0,-1), (1,0,-1) \rangle} = \sqrt{2}$

$\|v\| = \sqrt{\langle v, v \rangle} = \sqrt{\langle (-2,0,2), (-2,0,2) \rangle} = \sqrt{(-2)^2 + 0^2 + 2^2} = \sqrt{8}$

$= 2\sqrt{2}$

ALSO; $u + v = (-1, 0, 1)$

$\|u+v\| = \sqrt{\langle u+v, u+v \rangle} = \sqrt{\langle (-1,0,1), (-1,0,1) \rangle}$

$= \sqrt{1+1} = \sqrt{2}$

$\therefore$ $\|u+v\| \neq \|u\| + \|v\|$

Ex 9) If $x$ and $y$ are vectors in an I.P.S. $V(F)$, then show that $x=y$ iff $\langle x,z\rangle = \langle y,z\rangle$ for all $z \in V$.

Sol:- Let $x=y$

T.P $\langle x,z\rangle = \langle y,z\rangle$

Now $\langle x,z\rangle - \langle y,z\rangle = \langle x-y, z\rangle$

$= \langle 0, z\rangle = 0$ $\left\{ \because \langle x-y,z\rangle = \langle x,z\rangle - \langle y,z\rangle \right.$

$\Rightarrow \langle x,z\rangle - \langle y,z\rangle = 0$

$\Rightarrow \langle x,z\rangle = \langle y,z\rangle$ $\left\{ \because \text{given } x=y,\ x-y=0 \right.$

Conversely: let $\langle x,z\rangle = \langle y,z\rangle$

R.T.P $x=y$

given $\langle x,z\rangle = \langle y,z\rangle$

$\Rightarrow \langle x,z\rangle - \langle y,z\rangle = 0 \Rightarrow \langle x-y,z\rangle = 0.$

$\Rightarrow \langle x-y,z\rangle = 0 \quad \forall\ z \in V.$

In particular; Take $z = x-y$ $\left\{ \because x,y \in V(F) \Rightarrow x-y \in V(F),\ z = x-y \in V(F) \right.$

$\Rightarrow \langle x-y, x-y\rangle = 0$

$\Rightarrow \langle u,u\rangle = 0$ where $u = x-y$

$\Leftrightarrow u = 0 \quad \Rightarrow x-y=0$

$\Rightarrow \boxed{x=y}$

Thm:- Prove that Every I.P.S. is a Metric Space.

Proof:- Let $V(F)$ be an I.P.S.

Let $\alpha, \beta \in V$ be any two arbitrary elts

Define $d(\alpha,\beta) = \|\alpha - \beta\|$

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T.P 'd' is a M.S. on V.

ie (i) $d(\alpha,\beta) \geq 0 \quad \forall\, \alpha,\beta \in V$

and $d(\alpha,\beta) = 0$ iff $\alpha = \beta$

(ii) $d(\alpha,\beta) = d(\beta,\alpha)$

(iii) $d(\alpha,\beta) \leq d(\alpha,\gamma) + d(\gamma,\beta)$ ; $\alpha,\beta,\gamma \in V$

(i) $d(\alpha,\beta) = \|\alpha-\beta\|$. [By def.]

$\geq 0$. {∵ Norm is always Non-negative}

Also $d(\alpha,\beta) = 0$

iff $\|\alpha-\beta\| = 0 \iff \sqrt{\langle \alpha-\beta, \alpha-\beta \rangle} = 0$

$\iff \langle \alpha-\beta, \alpha-\beta \rangle = 0$

$\iff \langle u,u \rangle = 0$ where $u = \alpha-\beta$

$\iff u = 0 \Rightarrow \boxed{\alpha = \beta}$

(ii) $d(\alpha,\beta) = \|\alpha-\beta\| = \|(-1)(\beta-\alpha)\|$

$= |(-1)|\,\|\beta-\alpha\|$ [∵ $\|\alpha u\| = |\alpha|\,\|u\|$]

$= 1 \cdot \|\beta-\alpha\|$

$= d(\beta,\alpha)$.

(iii) Let $\alpha, \beta, \gamma \in V$.

Then $d(\alpha,\gamma) + d(\gamma,\beta) = \|\alpha-\gamma\| + \|\gamma-\beta\|$

$\geq \|\alpha-\gamma+\gamma-\beta\|$ [Using triangle inequality $\|x+y\| \leq \|x\|+\|y\|$]

$\geq \|\alpha-\beta\|$

$\geq d(\alpha,\beta)$

$\Rightarrow d(\alpha,\beta) \leq d(\alpha,\gamma) + d(\gamma,\beta)$

∴ 'd' satisfies all properties of a M.S.

So; V is a M.S.